

# Equations of motion approach to the spin-1/2 Ising model on the Bethe lattice

Ferdinando Mancini\* and Adele Naddeo†

*Dipartimento di Fisica "E. R. Caianiello" - Unità CNISM di Salerno,  
Università degli Studi di Salerno, 84081 Baronissi (SA), Italy*

(Dated: February 1, 2008)

We exactly solve the ferromagnetic spin-1/2 Ising model on the Bethe lattice in the presence of an external magnetic field by means of the equations of motion method within the Green's function formalism. In particular, such an approach is applied to an isomorphic model of localized Fermi particles interacting via an intersite Coulomb interaction. A complete set of eigenoperators is found together with the corresponding eigenvalues. The Green's functions and the correlation functions are written in terms of a finite set of parameters to be self-consistently determined. A procedure is developed, that allows us to exactly fix the unknown parameters in the case of a Bethe lattice with any coordination number  $z$ . Non-local correlation functions up to four points are also provided together with a study of the relevant thermodynamic quantities.

PACS numbers: 05.50.+q, 05.30.Fk, 75.10.-b

## I. INTRODUCTION

Recently, it has been shown [1] that a system built up of  $q$  species of Fermi particles, localized on the sites of a Bravais lattice and subjected to finite-range interactions, is exactly solvable in any dimension. Exactly solvable means that it is always possible to find a complete set of eigenvalues and eigenoperators of the Hamiltonian, which close the hierarchy of the equations of motion. In such a way, exact expressions for the relevant Green's functions and correlation functions can be derived. These expressions are just formal because they depend on a finite set of parameters to be self-consistently determined. In Refs. [2, 3, 4] it has been shown how it is possible to fix such parameters exactly by means of algebra constraints in the case of one dimension and  $q = 1$ ,  $q = 2$  and  $q = 3$ , respectively. In this way, complete and exact solutions of these systems have been obtained.

A system of  $q$  species of Fermi particles has been shown [1] to be isomorphic to a spin- $\frac{q}{2}$  Ising-like model in the presence of an external magnetic field, so opening a different route to the study of spin systems, which can be very difficult in two and three dimensions when attacked by the transfer matrix method. Furthermore, this approach can shed new light on how to get an exact solution for these systems in higher dimensions in the presence of an external magnetic field as it is always possible [1] to find an exact expression for the corresponding Green's functions and correlation functions. The exact knowledge of the eigenenergies of the system can give information on the energy scales ruling the physical behavior and the response of the system and can find an application as unbiased check for the approximate studies present in the literature. Within our approach the problem is that correlation and Green's functions depend on a finite set of unknown parameters to be self-consistently determined. A complete exact solution of the system is obtained only when such parameters are known. These parameters cannot be determined by means of the dynamics and are fixed by choosing the representation where the field operators are realized. In particular [cfr. Section 2.4 in Ref. [6]], they can be fixed by appropriate self-consistent equations which are the manifestation of symmetries of the model, algebraic properties of the field operators, boundary conditions (i.e. properties of the underlying lattice, phase of the system according to the values of the external thermodynamical parameters). It has been shown how to fix exactly such parameters by means of algebra constraints in the one dimensional (1D) case [2, 3, 4] and we are now working on the possibility to introduce new algebra constraints and topological relations in order to fix the self-consistent parameters in higher dimensions. The first step to realize such an ambitious program appears to be the application of our technique to a Bethe lattice of any coordination number  $z$ . In this article we apply our formulation, the Composite Operator Method (COM) [5, 6], to the study of a system of  $q = 1$  species of Fermi particles interacting with an intersite Coulomb interaction on the Bethe lattice with coordination number  $z$ . Such a model is shown to be isomorphic to the spin- $\frac{1}{2}$  Ising model on the Bethe lattice in the presence of an external magnetic field.

Bethe [7] and Bethe-like lattices [8] have been widely studied in solid state and statistical physics because they

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\*Electronic address: mancini@sa.infn.it

†Electronic address: naddeo@sa.infn.it

represent the underlying lattices on which many systems can be exactly solved [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. Following the line of reasoning that refers to the mean field theory of magnetism, the Bethe-Peierls [22] approximation was developed in 1935/36 in order to describe crystalline alloys or Ising models. It takes exactly into account the interaction of a given spin with its nearest neighbors but introduces a mean field in order to express the interactions between such neighbors and all the other spins in the lattice. Later, it was pointed out [23] that such an approximation becomes exact on the Bethe lattice. In particular, it was shown that the partition function of the ferromagnetic spin- $\frac{1}{2}$  Ising model on the Bethe lattice with any coordination number  $z$  is equivalent to that in the Bethe approximation [23]. Then, the equivalence of the exact solution of the Bethe lattice spin- $\frac{1}{2}$  Ising model to the Bethe-Peierls approximation was shown also in the antiferromagnetic case [24]. Summarizing, there are two special properties that make Bethe lattices particularly suited for theoretical investigations: the self-similar structure which may lead to recursive solutions and the absence of closed loops which restricts interference effects of quantum-mechanical particles in the case of nearest-neighbor coupling. Furthermore, Bethe and Bethe-like lattices have attracted a lot of interest because they usually reflect essential features of systems even when conventional mean-field theories fail [17]. The reason is that such lattices are capable to take into account correlations which are usually lost in conventional mean-field calculations. The spin- $\frac{1}{2}$  Ising model on the Bethe lattice can be exactly solved by means of the transfer matrix technique [25] which reduces the solution to an eigenvalue problem of the second order and all the relevant thermodynamic quantities such as the magnetization and the susceptibility, and so on, can be calculated by means of recursion relations, thanks to the nested structure of the underlying lattice. The same technique has been recently employed in the exact calculation of the spin-spin correlation functions  $\langle S(0)S(n) \rangle$  for any temperatures  $T$  and external field  $h$  [26, 27]. Exact expressions for the free energy and the magnetization of a spin- $\frac{1}{2}$  Ising model on a two-layer Bethe lattice in the presence of magnetic fields different in the two layers have been also obtained together with a study of the whole phase diagram by means of an iteration technique [28]. Also, it should be mentioned a large activity in the framework of the athermal random-field Ising model (RFIM), where analytical results have been obtained on Bethe lattices [29, 30].

In this paper we exactly solve the ferromagnetic spin- $\frac{1}{2}$  Ising model on the Bethe lattice with any coordination number  $z$  in the presence of an external magnetic field within the COM approach [5, 6]. All the Green's functions and correlation functions are obtained together with the behavior of the relevant thermodynamic properties. Two-point  $\langle S(0)S(j) \rangle$  and three-point  $\langle S(0)S(j)S(k) \rangle$  spin-spin correlation functions are also provided together with non local correlation functions of higher order. The manuscript is organized as follows. In Section 2, we give the general Hamiltonian of the spin- $\frac{1}{2}$  Ising model on the Bethe lattice and the mapping onto a model of Fermi particles with intersite Coulomb interactions. In Section 3, we present the general solution in terms of eigenvalues and eigenvectors. In Section 4, we show how to close the system of self-consistent equations and find the unknown parameters in order to compute the correlation and the Green's functions. In Section 5, we compute the local correlation functions and in Section 6, the non-local ones. In Section 7, we study all the relevant thermodynamic quantities, such as magnetization, susceptibility, internal energy, specific heat and entropy as functions of the temperature and the external magnetic field, specializing the general formulas to the case of a Bethe lattice with coordination number  $z = 3$  and  $z = 4$ . Finally, some concluding remarks and outlooks of our work are given. Some technical Appendices follow.

As a final remark, we would like to stress that the motivation of this work is to show that the formalism of Green's functions and equations of motion is a convenient technique to study spin systems. Most of the techniques used in the literature for the study of these systems are based on the transfer matrix method. This latter formalism is a very powerful technique and has been largely applied with success to a huge number of models. After the brilliant solution by Onsager [31] for the two-dimensional spin- $\frac{1}{2}$  Ising model in zero field, many other two-dimensional (2D) models, such as the dimer problem, six-vertex, eight-vertex (see Baxter's book), have been solved by making use of the transfer matrix method [see Baxter's book [25] for a comprehensive list of references]. However, it should be noticed that this method is very transparent and convenient for the case of one dimension, but becomes complicate for higher dimensions. In spite of the tremendous work done [among the most recent results, the derivation of the order parameter of the chiral Potts model by Baxter [32] has to be mentioned], many problems remain unsolved. The exact partition function in a finite magnetic field is still unknown. No exact results have been obtained for the three-dimensional model. By using the equation of motion formalism, we have constructed a general method to study Ising spin systems [1]. Such a method is general, in the sense that it has been formulated for any dimension of the system. We can exactly calculate a complete set of eigenoperators and eigenvalues of the Hamiltonian, and consequently to derive analytical expressions for the correlation functions. In order that this scheme of calculation could be used in practice, it is necessary to calculate a set of unknown parameters. The number of unknown parameters depends not only on the dimensions of the system, but also on the dimension of the spin; for a Ising spin- $q/2$  system on a lattice of coordination number  $z$ , the number of unknown parameters is  $2qz$  [cfr. Ref. [1]]. Our previous studies [1, 2, 3, 4] show that it is possible to find the necessary self-consistent equations by using not only properties of the lattice, but also symmetry and algebraic properties of the field operators. In the last two years we have been performing a systematic study of this last point. We started by considering the simplest problem of spin-1/2 on a linear chain [2]. Then, we

considered the case of spin-1 [3] and spin-3/2 [4], always for 1D systems. The extension to spin higher than 1/2 is not immediate, but requires the introduction of higher composite fields (projection operators). After this study of 1D systems, we decided to consider more complicate lattices, by considering the Bethe lattice. This lattice has the same topology of 1D because the absence of closed loops, but the analysis requires a dependence on the coordination number. For the Bethe lattice we have shown that the problem can be completely solved; we have shown that all the known results existing in literature can be reproduced. Furthermore, we have obtained new results, not previously obtained, as for the case of three-point correlation functions. The next step we have in program is the study of the spin-1/2 for the 2D lattice. This step is a very hard task; the properties of the lattice are different and the introduction of new concepts for writing down the self-consistent equations for the unknown parameters will be necessary.

## II. THE MODEL

Let us consider the spin- $\frac{1}{2}$  Ising model with nearest-neighbor interactions, in presence of an uniform external magnetic field  $h$ , on a Cayley tree with coordination number  $z$ . The Hamiltonian can be written as:

$$H = -hS(0) + \sum_{p=1}^z H^{(p)} \quad (1)$$

where  $S(0)$  is the spin operator at the central site (0). The spin variables  $S$  take only two values:  $S = \pm 1$ .  $H^{(p)}$  is the Hamiltonian of the  $p$ -th sub-tree rooted at the site (0) and can be written as:

$$H^{(p)} = -hS(p) - JS(0)S(p) + \sum_{m=1}^{z-1} H^{(p,m)} \quad (2)$$

where  $(p)$ ,  $(p = 1, \dots, z)$  are the nearest neighbors of (0), also termed the first shell. In turn  $H^{(p,m)}$  describes the  $m$ -th sub-tree rooted at the site  $(p)$ . The process may be continued until we eventually reach the boundary sites, described by the Hamiltonian:

$$H^{(p_1, \dots, p_r)} = -hS(p_1, \dots, p_r) - JS(p_1, \dots, p_{r-1})S(p_1, \dots, p_r), \quad (3)$$

where  $(p_1, p_2, \dots, p_r)$  [ $p_1 = 1, \dots, z$ ;  $p_2, p_3, \dots = 1, \dots, z-1$ ] are the boundary points belonging to the  $r$ -th shell. In what follows we focus only on the sites deep in the interior of the tree, so ignoring the boundary, i. e. we concentrate on the Bethe lattice. Let us now consider the transformation:

$$S(i) = 2n(i) - 1 \quad (4)$$

where  $i$  is a generic site of the lattice,

$$n(i) = c^\dagger(i)c(i) \quad (5)$$

is the density operator for a spinless fermionic field,  $c(i)$  and  $c^\dagger(i)$  being the annihilation and creation operators satisfying the canonical anti-commutation relations:

$$\begin{aligned} \{c(\mathbf{i}, t), c^\dagger(\mathbf{j}, t)\} &= \delta_{\mathbf{i}\mathbf{j}} \\ \{c(\mathbf{i}, t), c(\mathbf{j}, t)\} &= \{c^\dagger(\mathbf{i}, t), c^\dagger(\mathbf{j}, t)\} = 0 \end{aligned} \quad (6)$$

In this way a mapping is established between the spin- $\frac{1}{2}$  Ising model and a model of Fermi particles with intersite Coulomb interactions on the Bethe lattice, where the correspondence between the Ising and the fermionic variables is:

$$\begin{aligned} S = 1 &\Rightarrow n = 1 \\ S = -1 &\Rightarrow n = 0 \end{aligned} \quad (7)$$

The Ising Hamiltonian, eqs. (1)-(3), with the transformation (4), takes the form:

$$\begin{aligned} H &= E_0 + 2(zJ - h)n(0) + \sum_{p=1}^z \hat{H}^{(p)} \\ \hat{H}^{(p)} &= 2(zJ - h)n(p) - 4Jn(0)n(p) + \sum_{m=1}^{z-1} \hat{H}^{(p,m)} \\ &\vdots \\ \hat{H}^{(p_1, \dots, p_r)} &= 2(zJ - h)n(p_1, \dots, p_r) - 4Jn(p_1, \dots, p_{r-1})n(p_1, \dots, p_r) \end{aligned} \quad (8)$$

where the constant term  $E_0$  is defined as:

$$E_0 = h + z(h - J) \sum_{p=1}^r (z - 1)^{p-1} = h + z(h - J) \frac{(z - 1)^r - 1}{z - 2}. \quad (9)$$

We immediately recognize the chemical potential  $\mu = 2(h - zJ)$  and the potential strength  $V = -4J$  in a fermionic language. Also here we ignore the boundary sites and reduce to the Bethe lattice. Such an Hamiltonian enjoys the particle-hole symmetry, that is, it turns out to be invariant under the transformation  $n \rightarrow 1 - n$ , which in the spin language corresponds to the spin-inversion symmetry  $S \rightarrow -S$ ,  $h \rightarrow -h$ ; in particular the chemical potential as a function of  $n$  scales as

$$\mu(1 - n) = zV - \mu(n). \quad (10)$$

This scaling law implies that the magnetization vanishes in zero external magnetic field. However, as it will be shown in Section 7 the Hamiltonian (1) and/or (8) admits also solutions exhibiting a spontaneous breakdown of the particle-hole symmetry; that is a magnetization different from zero in absence of magnetic field.

We see that the density operator satisfies the equation of motion:

$$i \frac{\partial}{\partial t} n(i) = [n(i), H] = 0, \quad (11)$$

so that standard methods based on the use of equations of motion and Green's function formalism are not immediately applicable in terms of this operator. The relevant equation of motion to be considered is:

$$i \frac{\partial}{\partial t} c(i) = -\mu c(i) - 4zJc(i)n^\alpha(i) \quad (12)$$

where

$$n^\alpha(i) = \frac{1}{z} \sum_{p=1}^z n(i, p), \quad (13)$$

$(i, p)$  being the nearest neighbors of the site  $i$ .

In the next Section, we will show in detail how to deal with such an issue and build up the formalism. We shall put the attention to the fermionic system and will solve the Hamiltonian (8) by using the formalism of the equation of motion and Green's function method [1, 2, 3, 4, 5, 6].

### III. COMPOSITE OPERATORS AND EQUATIONS OF MOTION

In this Section, we exactly solve the Hamiltonian (8) starting from the identification of a suitable operatorial basis [5, 6]. In order to pursue this task, we focus on the central site (0), even though we could have chosen any other site thanks to the symmetry of the Bethe lattice. Let us consider the following series of composite field operators

$$\psi_k(0) = c(0) [n^\alpha(0)]^{k-1}, \quad k = 1, 2, \dots \quad (14)$$

where, according to the definition (13)  $n^\alpha(0) = \frac{1}{z} \sum_{p=1}^z n(0, p)$ ,  $(p)$  being the first neighbors of the site (0). By using (11) and (12) it is easy to see that these operators satisfy the hierarchy of equations of motion:

$$i \frac{\partial}{\partial t} \psi_k(0) = [\psi_k(0), H] = -\mu \psi_k(0) - 4zJ \psi_{k+1}(0). \quad (15)$$

However, we observe that the number operator  $n(i)$  satisfies the following algebra:

$$[n(i)]^k = [c^\dagger(i) c(i)]^k = n(i), \quad k = 1, 2, \dots \quad (16)$$

As shown in Appendices A and B, this algebraic property allows us to establish the following fundamental property of the fields  $[n^\alpha(i)]^k$

$$[n^\alpha(i)]^k = \sum_{m=1}^z A_m^{(k)} [n^\alpha(i)]^m \quad (17)$$

where the coefficients  $A_m^{(k)}$  are rational numbers which can be calculated according to the scheme given in Appendix B. Therefore, for  $k = z + 1$  the hierarchy of equations of motion (15) closes as the additional operator  $\psi_{z+2}(0) = c(0)[n^\alpha(0)]^{z+1}$  can be rewritten in terms of the previous  $z + 1$  elements of (14) through the relation (17). We are thus able to derive a closed set of eigenoperators of the Hamiltonian by defining the following composite operator:

$$\psi(0) = \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \\ \vdots \\ \psi_{z+1}(0) \end{pmatrix} = \begin{pmatrix} c(0) \\ c(0)n^\alpha(0) \\ \vdots \\ c(0)[n^\alpha(0)]^z \end{pmatrix} \quad (18)$$

which satisfies the equation of motion:

$$i\frac{\partial}{\partial t}\psi(0) = [\psi(0), H] = \varepsilon\psi(0), \quad (19)$$

where the  $(z + 1) \times (z + 1)$  energy matrix  $\varepsilon$  is defined as:

$$\varepsilon = \begin{pmatrix} -\mu & -4zJ & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\mu & -4zJ & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\mu & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\mu & -4zJ & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\mu & -4zJ \\ 0 & -4zJA_1^{(z+1)} & -4zJA_2^{(z+1)} & \cdots & -4zJA_{z-2}^{(z+1)} & -4zJA_{z-1}^{(z+1)} & -\mu - 4zJA_z^{(z+1)} \end{pmatrix}. \quad (20)$$

The eigenvalues  $E_n$  of the energy matrix have the expressions

$$E_n = -\mu - 4(n - 1)J, \quad n = 1, 2, \dots, z + 1. \quad (21)$$

At this stage we can say that we have formally, but exactly, solved Hamiltonian (8) or its spin counterpart (1)-(3) as we have found for them a complete set of eigenoperators and eigenvalues for any coordination number  $z$  of the underlying Bethe lattice. The solution is formal as we have to compute still the correlation functions.

In order to do this, let us now define the thermal retarded Green's function

$$G^R(t - t') = \langle R[\psi(0, t)\psi^\dagger(0, t')] \rangle = \theta(t - t') \langle \{\psi(0, t), \psi^\dagger(0, t')\} \rangle \quad (22)$$

where  $\langle \dots \rangle$  denotes the quantum-statistical average over the grand canonical ensemble. By introducing the Fourier transform:

$$G^R(t - t') = \frac{i}{(2\pi)} \int_{-\infty}^{+\infty} d\omega e^{-i\omega(t-t')} G^R(\omega) \quad (23)$$

and by means of the Heisenberg equation (19) we get the equation:

$$[\omega - \varepsilon] G^R(\omega) = I \quad (24)$$

where  $I$  is the normalization matrix, defined as:

$$I = \langle \{\psi(0, t), \psi^\dagger(0, t)\} \rangle. \quad (25)$$

The solution of Eq. (24) is [5, 6]

$$G^R(\omega) = \sum_{n=1}^{z+1} \frac{\sigma^{(n)}}{\omega - E_n + i\delta} \quad (26)$$

where  $\sigma^{(n)}$  are the spectral density matrices, to be calculated through the formula [5, 6]:

$$\sigma_{ab}^{(n)} = \Omega_{an} \sum_{c=1}^{z+1} [\Omega_{nc}]^{-1} I_{cb}, \quad (27)$$

where  $\Omega$  is the  $(z+1) \times (z+1)$  matrix whose columns are the eigenvectors of the matrix  $\varepsilon$ .

The matrix  $\Omega$  has the expression

$$\Omega = \begin{pmatrix} 1 & z^z & \left(\frac{z}{2}\right)^z & \cdots & \left(\frac{z}{z-2}\right)^z & \left(\frac{z}{z-1}\right)^z & 1 \\ 0 & z^{z-1} & \left(\frac{z}{2}\right)^{z-1} & \cdots & \left(\frac{z}{z-2}\right)^{z-1} & \left(\frac{z}{z-1}\right)^{z-1} & 1 \\ 0 & z^{z-2} & \left(\frac{z}{2}\right)^{z-2} & \cdots & \left(\frac{z}{z-2}\right)^{z-2} & \left(\frac{z}{z-1}\right)^{z-2} & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & z^2 & \left(\frac{z}{2}\right)^2 & \cdots & \left(\frac{z}{z-2}\right)^2 & \left(\frac{z}{z-1}\right)^2 & 1 \\ 0 & z^1 & \left(\frac{z}{2}\right)^1 & \cdots & \left(\frac{z}{z-2}\right)^1 & \left(\frac{z}{z-1}\right)^1 & 1 \\ 0 & z^0 & \left(\frac{z}{2}\right)^0 & \cdots & \left(\frac{z}{z-2}\right)^0 & \left(\frac{z}{z-1}\right)^0 & 1 \end{pmatrix}; \quad (28)$$

in general, the matrix element  $\Omega_{p,k}$  has the expression:

$$\Omega_{p,k} = \begin{cases} 1 & k=1, p=1 \\ 0 & k=1, p \neq 1 \\ \left(\frac{z}{k-1}\right)^{z+1-p} & k \neq 1 \end{cases}. \quad (29)$$

By means of the definition (25) and of the recursion rule (17), the normalization matrix can be easily calculated and has the expression

$$I = \begin{pmatrix} 1 & I_{1,2} & I_{1,3} & \cdots & I_{1,z-1} & I_{1,z} & I_{1,z+1} \\ I_{1,2} & I_{1,3} & I_{1,4} & \cdots & I_{1,z} & I_{1,z+1} & I_{2,z+1} \\ I_{1,3} & I_{1,4} & I_{1,5} & \cdots & I_{1,z+1} & I_{2,z+1} & I_{3,z+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ I_{1,z-1} & I_{1,z} & I_{1,z+1} & \cdots & I_{z-3,z+1} & I_{z-2,z+1} & I_{z-1,z+1} \\ I_{1,z} & I_{1,z+1} & I_{2,z+1} & \cdots & I_{z-2,z+1} & I_{z-1,z+1} & I_{z,z+1} \\ I_{1,z+1} & I_{2,z+1} & I_{3,z+1} & \cdots & I_{z-1,z+1} & I_{z,z+1} & I_{z+1,z+1} \end{pmatrix} \quad (30)$$

where the elements  $I_{p,z+1}$  ( $p = 2, \dots, z+1$ ) are expressed as

$$I_{p,z+1} = \sum_{m=1}^z A_m^{(p+z-1)} I_{1,m+1}. \quad (31)$$

Therefore we need to know only the elements  $I_{1,k}$  ( $k = 2, \dots, z+1$ ) which are given by

$$I_{1,k} = \langle [n^\alpha(0)]^{k-1} \rangle = \kappa^{(k-1)}. \quad (32)$$

Then, the spectral density matrices  $\sigma^{(n)}$  can be easily calculated by means of Eq. (27) once we keep in mind that, according to the structure of the normalization matrix  $I$ , Eqs. (30) and (31), there exist only  $z+1$  independent matrix elements  $\sigma_{1,k}^{(n)}$  for each of the  $z+1$  matrices  $\sigma^{(n)}$  while all the others can be obtained as linear combinations of these latter according to (31). As a result we get:

$$\sigma^{(n)} = \Sigma_n \Gamma^{(n)} \quad (33)$$

where  $\Sigma_n$  are functions of the elements  $I_{1,k}$  with  $k = 1, \dots, z+1$  and  $\Gamma^{(n)}$  are numerical matrices. In particular we have the following expressions:

$$\Sigma_p = \left(\frac{z}{p-1}\right)^z \sum_{k=2}^{z+1} \Omega_{p,k}^{-1} I_{1,k}, \quad p = 2, \dots, z+1 \quad (34)$$

and

$$\begin{aligned} \Gamma_{1,k}^{(1)} &= (1, 0, 0, \dots, 0, 0, 0) \quad k = 1, \dots, z+1 \\ \Gamma_{1,k}^{(n)} &= \left(\frac{n-1}{z}\right)^{k-1}, \quad n = 2, \dots, z+1 \end{aligned} \quad (35)$$

The correlation function

$$C(t-t') = \langle \psi(0, t) \psi^\dagger(0, t') \rangle = \frac{1}{(2\pi)} \int_{-\infty}^{+\infty} d\omega e^{-i\omega(t-t')} C(\omega) \quad (36)$$

can be computed starting from Eq. (26) and recalling the relation:

$$C(\omega) = - \left[ 1 + \tanh \left( \frac{\beta\omega}{2} \right) \right] \Im(G^R(\omega)). \quad (37)$$

Then we get:

$$C(\omega) = \pi \sum_{n=1}^{z+1} \sigma^{(n)} T_n \delta[\omega - E_n] \quad (38)$$

$$C(t-t') = \frac{1}{2} \sum_{n=1}^{z+1} e^{-iE_n(t-t')} \sigma^{(n)} T_n \quad (39)$$

where

$$T_n = 1 + \tanh \left( \frac{\beta E_n}{2} \right). \quad (40)$$

Eqs. (26) and (38) are the exact solution of the problem. Such a solution is only formal because the complete knowledge of the retarded and correlation functions is not fully achieved owing to the presence of the unknown static correlation functions  $\kappa^{(m)} = \langle [n^\alpha(0)]^m \rangle$  ( $m = 1, \dots, z$ ) appearing in the normalization matrix  $I$  due to the non-canonical algebra satisfied by the composite field operator  $\psi(0)$ . Such unknown parameters will be calculated according to the self-consistent scheme given in the following Section.

#### IV. SELF-CONSISTENCY

As we have shown in the previous Section, a complete solution of the model requires the knowledge of the correlators  $\kappa^{(m)} = \langle [n^\alpha(0)]^m \rangle$ . In order to compute these quantities, let us write the Hamiltonian (8) as the sum of two commuting terms:

$$\begin{aligned} H &= H_0 + H_I \\ H_I &= -4Jn(0) \sum_{p=1}^z n(p). \end{aligned} \quad (41)$$

Because  $[H_0, H_I] = 0$ , for any operator  $O$  we can write its average as

$$\langle O \rangle = \frac{\text{Tr} \{ O e^{-\beta H} \}}{\text{Tr} \{ e^{-\beta H} \}} = \frac{\text{Tr} \{ O e^{-\beta H_I} e^{-\beta H_0} \}}{\text{Tr} \{ e^{-\beta H_I} e^{-\beta H_0} \}} = \frac{\langle O e^{-\beta H_I} \rangle_0}{\langle e^{-\beta H_I} \rangle_0} \quad (42)$$

where  $\langle \dots \rangle_0$  is the trace with respect to the reduced Hamiltonian  $H_0$

$$\langle \dots \rangle_0 = \frac{\text{Tr} \{ \dots e^{-\beta H_0} \}}{\text{Tr} \{ e^{-\beta H_0} \}}. \quad (43)$$

Let us now consider the correlation functions  $C_{1,k} = \langle c(0) c^\dagger(0) [n^\alpha(0)]^{k-1} \rangle$ ,  $k = 1, \dots, z+1$ . By means of Eq. (42) we can derive the following relation:

$$\frac{C_{1,k}}{C_{1,1}} = \frac{\langle c(0) c^\dagger(0) [n^\alpha(0)]^{k-1} e^{-\beta H_I} \rangle_0}{\langle c(0) c^\dagger(0) e^{-\beta H_I} \rangle_0}, \quad k = 1, \dots, z+1. \quad (44)$$

Now from the Pauli principle we have the algebraic relation:

$$c^\dagger(i) n(i) = 0 \quad (45)$$

which leads to the property

$$c^\dagger(0) e^{-\beta H_I} = c^\dagger(0). \quad (46)$$

Then Eq. (44) takes the form

$$\frac{C_{1,k}}{C_{1,1}} = \frac{\langle c(0)c^\dagger(0)[n^\alpha(0)]^{k-1} \rangle_0}{\langle c(0)c^\dagger(0) \rangle_0}, \quad k = 1, \dots, z+1. \quad (47)$$

Now let us observe that  $H_0$  describes a system where the original lattice has been reduced to the central site (0) and to  $z$  sublattices, all disconnected among them and topologically equivalent to the starting one. Therefore, in the  $H_0$ -representation the correlation functions which connect sites belonging to disconnected graphs can be decoupled:

$$\begin{aligned} \langle f\{n(0)\} g\{n(p)\} \rangle_0 &= \langle f\{n(0)\} \rangle_0 \langle g\{n(p)\} \rangle_0 \\ \langle g\{n(p)\} h\{n(q)\} \rangle_0 &= \langle g\{n(p)\} \rangle_0 \langle h\{n(q)\} \rangle_0. \end{aligned} \quad (48)$$

Here  $f\{n(0)\}$ ,  $g\{n(p)\}$  and  $h\{n(q)\}$ , with  $p$  and  $q$  belonging to different sublattices, are any functions of the particle density. By means of such a property, Eq. (47) can be cast in the following form

$$\frac{C_{1,k}}{C_{1,1}} = \left\langle [n^\alpha(0)]^{k-1} \right\rangle_0, \quad k = 1, \dots, z+1. \quad (49)$$

In Appendix C we show that

$$\left\langle [n^\alpha(0)]^k \right\rangle_0 = F^{(z,k)}[X] \quad k = 1, \dots, z \quad (50)$$

where  $F^{(z,k)}[X]$  is a polynomial of order  $k$  in the variable  $X$ , defined as

$$X = \langle n^\alpha(0) \rangle_0 = \frac{C_{1,2}}{C_{1,1}}, \quad (51)$$

whose explicit expression is

$$F^{(z,k)}[X] = \sum_{p=1}^z a_p^{(z,k)} X^p, \quad a_p^{(z,k)} = \frac{1}{z^k} b_p^{(k)} \binom{z}{p}; \quad (52)$$

here  $b_p^{(k)}$  are some numerical coefficients defined in Appendix C. The previous analysis shows that all the properties of the system can be expressed in terms of only one parameter,  $X$ , defined by Eq. (51). In order to determine this parameter we use the self-consistent equation

$$C_{1,1} = 1 - \kappa^{(1)}, \quad (53)$$

where we required the translational invariant condition  $\langle n^\alpha(i) \rangle = \langle n(i) \rangle$ . In order to exploit this equation we note that from (39), by using the definition (27), we get:

$$I_{ab} = C_{a,b} + \sum_{m,p=1}^{z+1} \Omega_{am} [\Omega_{mp}]^{-1} e^{-\beta E_m} C_{p,b}. \quad (54)$$

Now, writing such equation for  $I_{11}$  and  $I_{12}$  and recalling that  $I_{11} = 1$ ,  $I_{12} = \kappa^{(1)}$ , we obtain

$$1 = C_{1,1} + \sum_{m,p=1}^{z+1} \Omega_{1m} [\Omega_{mp}]^{-1} e^{-\beta E_m} C_{p,1} \quad (55)$$

$$\kappa^{(1)} = C_{1,2} + \sum_{m,p=1}^{z+1} \Omega_{1m} [\Omega_{mp}]^{-1} e^{-\beta E_m} C_{p,2}. \quad (56)$$

Putting such expressions in the right hand side of the self-consistent equation (53) we get

$$C_{1,1} = C_{1,1} - C_{1,2} + \sum_{m,p=1}^{z+1} \Omega_{1m} [\Omega_{mp}]^{-1} e^{-\beta E_m} (C_{p,1} - C_{p,2}), \quad (57)$$



which, by using Eqs. (49) and (51), can be rewritten as

$$1 = (1 - X) + \sum_{m=1}^{z+1} e^{-\beta E_m} W^{(z,m)}, \quad (58)$$

where

$$W^{(z,m)} = \sum_{p=1}^{z+1} \Omega_{1m} [\Omega_{mp}]^{-1} \left( \langle [n^\alpha(0)]^{p-1} \rangle_0 - \langle [n^\alpha(0)]^p \rangle_0 \right). \quad (59)$$

The result (50) allows us to express the function  $W^{(z,m)}$  in terms of the parameter  $X$ ; indeed it can be shown that

$$W^{(z,m)} = \binom{z-1}{m-1} X^{m-1} (1-X)^{z+1-m} \quad m = 1, \dots, z \quad . \\ W^{(z,z+1)} = 0 \quad (60)$$

Then, it is possible to write Eq. (58) as follows:

$$1 = (1 - X) \left[ 1 + \sum_{m=1}^z e^{-\beta E_m} \binom{z-1}{m-1} X^{m-1} (1-X)^{z-m} \right]. \quad (61)$$

Recalling that  $E_m = -\mu - 4(m-1)J$ ,  $\mu = 2(h - zJ)$  and making some algebraic manipulations such equation takes finally the form:

$$X = (1 - X) e^{\beta\mu} [1 + (e^{4\beta J} - 1) X]^{z-1}. \quad (62)$$

Eq. (62) is the main result of this Section; it allows us to determine the parameter  $X$  in terms of the external parameters  $\mu$ ,  $T$ ,  $J$  (or  $h$ ,  $T$ ,  $J$ ).

Let us notice that, if we define a parameter  $x$  as

$$x = \frac{e^{2\beta J}}{1 + (e^{4\beta J} - 1) X}, \quad (63)$$

then Eq. (62) can be cast in the form

$$e^{2\beta h} = x^{z-1} \frac{(e^{2\beta J} - x)}{(xe^{2\beta J} - 1)} \quad (64)$$

which coincides with the one given by Baxter (see Ref. [25], p. 53). In our case the quantity  $X$  has a definite physical meaning, it is the particle density of the first shell in the  $H_0$ -representation.

## V. LOCAL CORRELATION FUNCTIONS AND RELATED PHYSICAL QUANTITIES

The aim of this Section is to compute all the local correlation functions by expressing them in terms of the parameter  $X$  introduced in Eq. (51). The calculation of the relevant physical quantities, that is particle density, magnetization, susceptibility, internal energy, specific heat and entropy per site, then easily follows.

Let us start by recalling the results (49), (50) and (52) which allow us to write the correlation functions in terms of the parameter  $X$  as follows

$$C_{1,k} = C_{1,1} \sum_{p=1}^z a_p^{(z,k-1)} X^p, \quad k = 2, \dots, z+1. \quad (65)$$

where  $C_{1,1}$ , due to Eq. (46), can be expressed as:

$$C_{1,1} = \frac{\langle c(0) c^\dagger(0) \rangle_0}{\langle e^{-\beta H_I} \rangle_0}. \quad (66)$$

In order to compute  $C_{1,1}$  let us observe that in the  $H_0$ -representation  $c(0)$  satisfies the equation of motion

$$i \frac{\partial}{\partial t} c(0) = -\mu c(0). \quad (67)$$

Then it is immediate to see that:

$$\langle c(0) c^\dagger(0) \rangle_0 = \frac{1}{e^{\beta\mu} + 1}, \quad \langle n(0) \rangle_0 = \frac{1}{e^{-\beta\mu} + 1}. \quad (68)$$

In order to evaluate the quantity  $\langle e^{-\beta H_I} \rangle_0$  let us observe that, by means of the algebraic property  $[n(i)]^m = n(i)$ , we can write

$$e^{-\beta H_I} = e^{4\beta J n(0) \sum_{p=1}^z n(p)} = \prod_{p=1}^z [1 + A n(0) n(p)] \quad (69)$$

where  $A = e^{4\beta J} - 1$ . By using the property (48) and by recalling that in the Bethe lattice all sites are equivalent and the parameter  $X$  satisfies Eq. (62), straightforward calculations show that:

$$\langle e^{-\beta H_I} \rangle_0 = 1 + [(1 + AX)^z - 1] \langle n(0) \rangle_0 = \frac{1 + AX^2}{(1 - X)(e^{\beta\mu} + 1)}. \quad (70)$$

Putting (68) and (70) into (66) we finally get

$$C_{1,1} = \frac{1 - X}{1 + AX^2}. \quad (71)$$

Now we are ready to calculate the particle density

$$n = \langle n(0) \rangle = 1 - C_{1,1} = \frac{X(1 + AX)}{1 + AX^2}, \quad (72)$$

the magnetization

$$m = \langle S(0) \rangle = 2 \langle n(0) \rangle - 1 = \frac{X(2 + AX) - 1}{1 + AX^2} \quad (73)$$

and all the correlation functions

$$C_{1,k} = \frac{1-X}{1+AX^2} \sum_{p=1}^z a_p^{(z,k-1)} X^p, \quad k = 2, \dots, z+1. \quad (74)$$

Let us now switch to the calculation of the correlation functions

$$\begin{aligned} \kappa^{(k)} &= \langle [n^\alpha(0)]^k \rangle \\ \lambda^{(k)} &= \langle n(0) [n^\alpha(0)]^k \rangle, \quad k = 1, \dots, z. \end{aligned} \quad (75)$$

According to the scheme given in Appendix D we have

$$\kappa^{(k)} = \frac{1}{1 + AX^2} \sum_{p=1}^z a_p^{(z,k)} X^p \left[ (1 - X) + \frac{X(1 + A)^p}{(1 + AX)^{p-1}} \right] \quad (76)$$

$$\lambda^{(k)} = \frac{1}{1 + AX^2} \sum_{p=1}^z a_p^{(z,k)} \frac{X^{p+1} (1 + A)^p}{(1 + AX)^{p-1}}. \quad (77)$$

The susceptibility per site can be calculated by means of Eq. (73) and has the expression:

$$\chi = \frac{\partial m}{\partial h} = \frac{\beta(1 - m^2)(1 + p)}{1 - (z - 1)p} \quad (78)$$

where we introduced the parameter  $p$ , defined as

$$p = \frac{AX(1 - X)}{1 + AX}. \quad (79)$$

This expression coincides with the one given in Refs. [26] with  $p$  playing the role of the ratio  $\gamma^{(0)}$  of the eigenvalues of the second order transfer matrix  $V$ .

Recalling the Hamiltonian (1), we obtain for the internal energy per site

$$E(T) = \frac{1}{N} \langle H \rangle = -J [m^2 (1-p) + p] - hm \quad (80)$$

where we used the fact that the total number of points in the graph is [25]

$$N = 1 + z \sum_{q=1}^r (z-1)^{q-1} = 1 + z \frac{(z-1)^r - 1}{z-2}. \quad (81)$$

Once  $E(T)$  is known, we can directly calculate the specific heat, the free energy and the entropy (per site) by means of the formulas:

$$C = \frac{dE}{dT}, \quad (82)$$

$$F(T) = E(T^*) - T \int_{T^*}^T \frac{E(\tilde{T}) - E(T^*)}{\tilde{T}^2} d\tilde{T}, \quad (83)$$

$$S(T) = \frac{E(T) - F(T)}{T}, \quad (84)$$

where the value of  $E$  is given by (80) and the limit  $T^* \rightarrow 0$  is understood.

## VI. NON LOCAL CORRELATION FUNCTIONS AND RELATED PHYSICAL QUANTITIES

In this Section we will calculate the relevant non local correlation functions; then we focus on the spin-spin one and on the related correlation length which we compare with the results existing in the literature [26]. We will show how our procedure allows us to evaluate also higher order non local functions with respect to the one given in Refs. [26]. Further technical details are presented in Appendix E.

### A. Two-point correlation functions

Let us start by defining the correlation functions

$$K^{(k)}(j) = \langle [n^\alpha(0)]^k n(j) \rangle \quad (85)$$

$$\Lambda^{(k)}(j) = \langle n(0) [n^\alpha(0)]^k n(j) \rangle \quad (86)$$

where  $j$  is a site at a distance of  $j$  steps from the central site. Let us make for simplicity the choice that  $j$  belongs to the  $z$ -th subtree (but any subtree can be chosen) and let us focus first on the two functions:

$$K^{(1)}(j) = \langle n^\alpha(0) n(j) \rangle \quad (87)$$

$$\Lambda^{(0)}(j) = \langle n(0) n(j) \rangle. \quad (88)$$

We see that  $\Lambda^{(0)}(j)$  is a two-point correlation function which connects two sites which are  $j$  steps apart. Observing that  $\langle n(z) n(j) \rangle$  and  $\langle n(p) n(j) \rangle_{p \neq z}$  connect two sites which are  $j-1$  and  $j+1$  steps apart, respectively, it is immediate to see that the two correlation functions  $K^{(1)}(j)$  and  $\Lambda^{(0)}(j)$  are related through the following relation

$$K^{(1)}(j) = \frac{1}{z} \Lambda^{(0)}(j-1) + \frac{z-1}{z} \Lambda^{(0)}(j+1). \quad (89)$$

Let us now study the function  $\Lambda^{(0)}(j)$ ; it is immediate to see that

$$\begin{aligned}\Lambda^{(0)}(0) &= \langle n(0) \rangle = n \\ \Lambda^{(0)}(1) &= \langle n(0)n(1) \rangle = \lambda^{(1)} = n^2 + n(1-n)p \\ \Lambda^{(0)}(2) &= \langle n(0)n(2) \rangle = \frac{1}{z-1} (z\kappa^{(2)} - n) = n^2 + n(1-n)p^2\end{aligned}\tag{90}$$

where we used the results of Appendices A and D [cfr. Eqs. (D10)-(D12)] and we noticed that by means of (D10) the parameter  $p$ , defined by Eq. (79), can be expressed as  $p = \frac{\lambda^{(1)} - n^2}{n(1-n)}$ . On the other hand, in Appendix E we prove the following recursion relation

$$G(j+1) - pG(j) = \frac{1}{p(z-1)} [G(j) - pG(j-1)]\tag{91}$$

where we defined

$$G(j) = \Lambda^{(0)}(j) - n^2.\tag{92}$$

Then, the two-point density correlation functions  $\Lambda^{(0)}(j)$  for any  $j$  take the expression

$$\Lambda^{(0)}(j) = n^2 + n(1-n)p^j \Rightarrow \frac{\Lambda^{(0)}(j) - n^2}{n(1-n)} = p^j.\tag{93}$$

We are now in the position to calculate higher order correlation functions. By putting (93) into (89) we get

$$K^{(1)}(j) = n^2 + \frac{1}{z}n(1-n)[p^{j-1} + (z-1)p^{j+1}],\tag{94}$$

while, by putting the result (93) into Eq. (E13) of Appendix E we obtain:

$$\Lambda^{(1)}(j) = n\lambda^{(1)} + \frac{(1-n)n}{z} \{np^{j-1} + [1+n(z-2)]p^j + (1-n)(z-1)p^{j+1}\}.\tag{95}$$

By noting that the parameter  $p$  can be written as

$$p = 1 - \frac{X}{n} = 1 - \frac{\langle n^\alpha(0) \rangle_0}{\langle n(0) \rangle}\tag{96}$$

we see that it is always  $p < 1$ . Then the correlation functions  $\Lambda^{(0)}(j)$ ,  $K^{(1)}(j)$ ,  $\Lambda^{(1)}(j)$  satisfy the ergodic theorem:

$$\begin{aligned}\lim_{j \rightarrow \infty} \Lambda^{(0)}(j) &= \langle n(0) \rangle \langle n(j) \rangle = n^2 \\ \lim_{j \rightarrow \infty} K^{(1)}(j) &= \langle n^\alpha(0) \rangle \langle n(j) \rangle = n^2 \\ \lim_{j \rightarrow \infty} \Lambda^{(1)}(j) &= \langle n(0)n^\alpha(0) \rangle \langle n(j) \rangle = n\lambda^{(1)}\end{aligned}\tag{97}$$

Recalling (4), we can evaluate from (93) the spin-spin correlation function

$$\langle S(0)S(j) \rangle = m^2 + (1-m^2)p^j.\tag{98}$$

This expression coincides with the result of Refs. [26]. Now, by defining the correlation function:

$$G_S(j) = \langle S(0)S(j) \rangle - \langle S(0) \rangle \langle S(j) \rangle\tag{99}$$

we obtain from (98)

$$G_S(j) = (1-m^2)e^{-\frac{j}{\xi}}\tag{100}$$

where the correlation length is defined as

$$\xi = \left[ \ln \left( \frac{1}{p} \right) \right]^{-1}.\tag{101}$$

### B. Three-point correlation functions

Following the same line of reasoning which led us to the two-point correlation functions, let us now calculate three-point correlation functions. Let us define the general three-point correlator as:

$$T^{(k)}(j, w) = \left\langle c(0) c^\dagger(0) [n^\alpha(0)]^{k-1} n(j) n(w) \right\rangle = M^{(k-1)}(j, w) - N^{(k-1)}(j, w) \quad (k \geq 1) \quad (102)$$

where we introduce the new correlation functions

$$M^{(k)}(j, w) = \left\langle [n^\alpha(0)]^k n(j) n(w) \right\rangle \quad (103)$$

$$N^{(k)}(j, w) = \left\langle n(0) [n^\alpha(0)]^k n(j) n(w) \right\rangle. \quad (104)$$

By  $j$  and  $w$  we denote two sites at a distance of  $j$  and  $w$  steps, respectively, with respect to the central site (0). Let us distinguish the two following cases: 1)  $j$  and  $w$  belong to the same subtree; 2)  $j$  and  $w$  belong to different subtrees.

#### Case 1

$j$  and  $w$  belong to the same subtree, which we take as the  $z$ -subtree, but any subtree can be chosen. By means of (42), (45) and (48) and by noting that:

$$\frac{\langle c(0) c^\dagger(0) \rangle_0}{\langle e^{-\beta H_I} \rangle_0} = C_{1,1} = 1 - n, \quad (105)$$

we can express  $T^{(k)}(j, w)$  as:

$$T^{(k)}(j, w) = M^{(k-1)}(j, w) - N^{(k-1)}(j, w) = (1 - n) \left\langle [n^\alpha(0)]^{k-1} n(j) n(w) \right\rangle_0. \quad (106)$$

Let us concentrate the attention on the two functions  $N^{(0)}(j, w) = \langle n(0) n(j) n(w) \rangle$  and  $M^{(1)}(j, w) = \langle n^\alpha(0) n(j) n(w) \rangle$ . At first, we note that these functions are related through the following relation

$$M^{(1)}(j, w) = \frac{1}{z} N^{(0)}(j-1, w-1) + \frac{z-1}{z} N^{(0)}(j+1, w+1). \quad (107)$$

Next, let us study the function  $N^{(0)}(j, w)$ ; by recalling the definitions (75) it is immediate to see that

$$\begin{aligned} N^{(0)}(0, 0) &= \langle n(0) n(0) n(0) \rangle = \langle n(0) \rangle = n \\ N^{(0)}(0, 1) &= N^{(0)}(1, 0) = N^{(0)}(1, 1) = \langle n(0) n(1) \rangle = \lambda^{(1)} \\ N^{(0)}(0, 2) &= N^{(0)}(2, 0) = N^{(0)}(2, 2) = \langle n(0) n(2) \rangle = \frac{1}{z-1} (z\kappa^{(2)} - n) \\ N^{(0)}(1, 2) &= N^{(0)}(2, 1) = \langle n(0) n(1) n(2) \rangle = \frac{1}{z-1} (z\lambda^{(2)} - \lambda^{(1)}) \end{aligned} \quad (108)$$

Recalling the relations (D12) in Appendix D and the expression  $p = \frac{\lambda^{(1)} - n^2}{n(1-n)}$ , the correlation function  $N^{(0)}(j, w)$  can be written in the closed form

$$N^{(0)}(j, w) = n^3 + n^2(1-n)(p^j + p^{w-j}) + n(1-n)^2 p^w, \quad j, w = 0, 1, 2. \quad (109)$$

In order to evaluate  $N^{(0)}(j, w)$  for all values of  $j$  and  $w$  we need a recursion formula as the one in Eq. (91), which we now derive following the same steps outlined in Appendix E for the function  $\Lambda^{(0)}(j)$ .

In the  $H_0$ -representation  $N^{(0)}(j, w)$  can be written as

$$N^{(0)}(j, w) = \langle n(0) n(j) n(w) \rangle = \frac{\langle n(0) n(j) n(w) e^{-\beta H_I} \rangle_0}{\langle e^{-\beta H_I} \rangle_0}. \quad (110)$$

Recalling that (cfr. (69))  $e^{-\beta H_I} = \prod_{p=1}^z [1 + A n(0) n(p)]$ , by making use of Eqs. (68) and (70), as well as of the equation (62) for the parameter  $X$ , we have

$$N^{(0)}(j, w) = \frac{X}{1 + AX^2} \langle n(j) n(w) (1 + A n(z)) \rangle_0. \quad (111)$$

Let us now calculate the function  $N^{(1)}(j, w) = \langle n(0) n^\alpha(0) n(j) n(w) \rangle$ . By following the same procedure, we obtain for  $j, w > 1$ :

$$N^{(1)}(j, w) = \frac{(z-1)}{z} \frac{X^2(1+A)}{(1+AX)(1+AX^2)} \langle n(j) n(w) (1+An(z)) \rangle_0 + \frac{1}{z} \frac{X(1+A)}{1+AX^2} \langle n(z) n(j) n(w) \rangle_0. \quad (112)$$

For simplicity, let us restrict the analysis to the case where  $w$  follows  $j$  (i. e. we can write  $w = j + r$ , where  $r$  is the number of steps necessary to go from  $w$  to  $j$ ). In this case we have  $M^{(0)}(j, w) = \Lambda^{(0)}(r)$  and Eq. (106) for  $k = 1$  gives

$$\Lambda^{(0)}(r) - N^{(0)}(j, j+r) = (1-n) \langle n(j) n(j+r) \rangle_0. \quad (113)$$

By combining (113) and (111) we can express the two quantities  $\langle n(j) n(j+r) \rangle_0$  and  $\langle n(j) n(j+r) n(z) \rangle_0$  in terms of  $N^{(0)}(j, j+r)$  and  $\Lambda^{(0)}(r)$  as follows:

$$\begin{aligned} \langle n(j) n(j+r) \rangle_0 &= \frac{1}{(1-n)} \left[ \Lambda^{(0)}(r) - N^{(0)}(j, j+r) \right] \\ \langle n(j) n(j+r) n(z) \rangle_0 &= \frac{1+AX^2}{AX(1-X)} N^{(0)}(j, j+r) - \frac{1}{A} \frac{1}{(1-n)} \Lambda^{(0)}(r). \end{aligned} \quad (114)$$

By using the relation (106) for  $k = 2$  and by observing that

$$\langle n^\alpha(0) n(j) n(j+r) \rangle_0 = \frac{1}{z} \langle n(j) n(j+r) n(z) \rangle_0 + \frac{z-1}{z} X \langle n(j) n(j+r) \rangle_0, \quad (115)$$

we can express the function  $M^{(1)}(j, j+r)$  in terms of  $N^{(0)}(j, j+r)$  and  $\Lambda^{(0)}(r)$  as

$$M^{(1)}(j, j+r) = \left[ \frac{(z-1)p}{z} + \frac{1}{zp} \right] N^{(0)}(j, j+r) + \left[ \frac{(z-1)p}{z} - \frac{1}{zp} \right] n(1-p) \Lambda^{(0)}(r). \quad (116)$$

Recalling that [see (93)]  $\Lambda^{(0)}(r) = n^2 + n(1-n)p^r$  we obtain from (107) the relevant recursion rule

$$\widehat{N}^{(0)}(j+1, w+1) - p\widehat{N}^{(0)}(j, w) = \frac{1}{p(z-1)} \left[ \widehat{N}^{(0)}(j, w) - p\widehat{N}^{(0)}(j-1, w-1) \right] \quad (117)$$

where we defined

$$\widehat{N}^{(0)}(j, w) = N^{(0)}(j, w) - n^3 - n^2(1-n)p^{w-j}. \quad (118)$$

By recalling the result (109), it is easy to see that the following expressions hold for any  $j$  and  $w$

$$\frac{\widehat{N}^{(0)}(j, w)}{n(1-n)} = np^j + (1-n)p^w \quad (119)$$

$$N^{(0)}(j, w) = n^3 + n^2(1-n)(p^j + p^{w-j}) + n(1-n)^2 p^w. \quad (120)$$

By using the transformation (4) we obtain from (120) the expression of the three-spin correlation function:

$$\langle S(0) S(j) S(w) \rangle = m^3 + m(1-m^2)(p^j - p^w + p^{w-j}); \quad (121)$$

this expression agrees with the one given in Ref. [33], where it was calculated for the one dimensional case (i. e.  $z = 2$ ). It is interesting to notice that the expressions of the spin correlation functions [cfr. (98) and (121)] depend on the coordination number  $z$  only through the parameters  $p$  and  $m$ .

We are now in position to calculate the correlation functions  $M^{(1)}(j, w)$  and  $N^{(1)}(j, w)$ . Straightforward calculations give

$$M^{(1)}(j, w) = n^3 + \frac{1}{z} n(1-n) \{ n(p^{j-1} + p^{w-j}) + (1-n)p^{w-1} + (z-1)[n(p^{j+1} + p^{w-j}) + (1-n)p^{w+1}] \}, \quad (122)$$

$$\begin{aligned} N^{(1)}(j, w) &= n^2 \lambda^{(1)} + n^2(1-n)[np^{w-j} + (1-n)p^{w-j+1}] + \\ &\quad \frac{n(1-n)}{z} [n^2 p^{j-1} + (1-n)^2 p^w + n(1-n)(p^{w-1} + p^j)] + \\ &\quad \frac{n(1-n)(z-1)}{z} [n^2 p^j + (1-n)^2 p^{w+1} + n(1-n)(p^{j+1} + p^w)]. \end{aligned} \quad (123)$$

Let us notice that all the correlators  $N^{(0)}(j, w)$ ,  $M^{(1)}(j, w)$  and  $N^{(1)}(j, w)$  satisfy the ergodic theorem.

## Case 2

Let us now switch to the second case:  $j$  and  $w$  belong to different subtrees, which we take as the  $z$ - and  $(z-1)$ -subtree, respectively. By performing the same steps which led to Eq. (106) and by noting that in such a case  $M^{(0)}(j, w) = \Lambda^{(0)}(j + w)$  we obtain the relation

$$N^{(0)}(j, w) = \Lambda^{(0)}(j + w) - (1 - n) \langle n(j) \rangle_0 \langle n(w) \rangle_0. \quad (124)$$

Recalling the expressions (E11) for  $\langle n(j) \rangle_0$  and (93) for  $\Lambda^{(0)}(j + w)$ , we obtain for any  $j$  and  $w$

$$N^{(0)}(j, w) = n^3 + n^2(1 - n)(p^j + p^w) + n(1 - n)^2 p^{j+w}. \quad (125)$$

To calculate higher order correlation functions we observe that

$$M^{(1)}(j, w) = \frac{1}{z} N^{(0)}(j - 1, w + 1) + \frac{1}{z} N^{(0)}(j + 1, w - 1) + \frac{z - 2}{z} N^{(0)}(j + 1, w + 1). \quad (126)$$

Putting (125) into (126) we have

$$\begin{aligned} M^{(1)}(j, w) = & n^3 + \frac{1}{z} n(1 - n) [n(p^{j-1} + p^{w-1}) + 2(1 - n)p^{j+w} + \\ & (z - 1)n(p^{j+1} + p^{w+1}) + (z - 2)(1 - n)p^{j+w+2}]. \end{aligned} \quad (127)$$

In order to calculate  $N^{(1)}(j, w)$  let us observe that

$$N^{(1)}(j, w) = M^{(1)}(j, w) - T^{(2)}(j, w) = M^{(1)}(j, w) - (1 - n) \langle n^\alpha(0) n(j) n(w) \rangle_0. \quad (128)$$

which, by noting that

$$\langle n^\alpha(0) n(j) n(w) \rangle_0 = \frac{1}{z} \langle n(z) n(j) \rangle_0 \langle n(w) \rangle_0 + \frac{1}{z} \langle n(z - 1) n(w) \rangle_0 \langle n(j) \rangle_0 + \frac{z - 2}{z} X \langle n(j) \rangle_0 \langle n(w) \rangle_0, \quad (129)$$

and recalling (E11) and (127), becomes

$$\begin{aligned} N^{(1)}(j, w) = & n^2 \lambda^{(1)} + \frac{[(z - 2)n + 1]}{z} n^2(1 - n)(p^j + p^w) + \frac{(z - 1)}{z} n^2(1 - n)^2(p^{j+1} + p^{w+1}) \\ & + \frac{n^3(1 - n)}{z} (p^{j-1} + p^{w-1}) + \frac{2}{z} n^2(1 - n)^2 p^{j+w-1} + \frac{(z - 2)}{z} n^3(1 - n) p^{j+w+1} \\ & + \frac{[2(1 - 2n) - (z - 4)n^2]}{z} n(1 - n) p^{j+w} + \frac{(z - 2)}{z} n(1 - n)^2 p^{j+w+2} \end{aligned} \quad (130)$$

for any  $j$  and  $w$ . Also in this case the ergodic theorem is satisfied.

## VII. RESULTS

In the previous Sections we have shown that all the properties of the system are expressed in terms of the correlator  $X = \langle n^\alpha(0) \rangle_0$ . This quantity is determined in terms of the external parameters  $J$ ,  $T$ ,  $h$  by solving the equation (62), which is a polynomial of order  $z$  in the variable  $X$ . In this Section we discuss the solutions of Eq. (62) and present the results obtained for various properties: the magnetization, the susceptibility, the specific heat, the free energy and the entropy. We shall discuss separately the cases of zero and finite magnetic field, by focusing the analysis to a ferromagnetic coupling (i. e.  $J > 0$ ).

### A. Zero magnetic field

In the case of zero magnetic field we have  $\mu = -2zJ$ ; then, it is useful to define  $K = e^{2\beta J}$  so that the equation (62) takes the form

$$X K^z = (1 - X) [1 + (K^2 - 1) X]^{z-1}. \quad (131)$$

It is easy to see that

$$X = \frac{1}{1+K} = \frac{1}{e^{2\beta J} + 1} \quad (132)$$

is always a solution of the equation (131) for any value of the coordination number  $z$ . By putting (132) into (72) and (73) we have

$$n = \langle n(0) \rangle = \frac{1}{2}, \quad m = \langle S(0) \rangle = 0. \quad (133)$$

The particle density and the magnetization do not depend on the temperature and on the coordination number  $z$ . This is a manifestation of the particle-hole symmetry, when we recall the scaling law (10) for the chemical potential. But (131) may admit other solutions which break the particle-hole symmetry. In particular, let us study if there is a critical temperature  $T_c$  such that the magnetization is different from zero for  $T < T_c$ . In order to determine  $T_c$  let us expand (131) in power series of  $X$  around the point given by (132). At first order we obtain

$$\left( X - \frac{1}{1+K} \right) [z(K-1) - 2K] = 0. \quad (134)$$

Therefore, besides the solution (132) there may be other solutions when  $K = \frac{z}{z-2}$ . Such an equation shows that there is a critical temperature  $T_c$ , given by

$$2J = k_B T_c \ln \left( \frac{z}{z-2} \right), \quad (135)$$

such that for  $T < T_c$  we may have solutions which spontaneously break the particle-hole symmetry and exhibit a magnetization different from zero. Let us notice that the case  $z = 2$  (i.e. the one-dimensional chain) gives  $T_c = 0$ . Let us also point out that Eq. (135) admits a solution only when  $J > 0$ . For negative  $J$  (i.e. antiferromagnetic coupling) there is no solution. If  $z$  is even, the equation (131) admits another solution

$$X = \frac{1}{1-K} = \frac{1}{1-e^{2\beta J}} \quad (136)$$

which also gives the results quoted in (133). However, such a solution describes an unstable system: the energy is a decreasing function of temperature and the parameter  $p$  is larger than one. This solution will be disregarded in the following.

Generally, for  $z > 2$  we have the following situation:

- $z$  even

$$\begin{cases} T < T_c & \begin{cases} 2 \text{ solutions corresponding to } n = \frac{1}{2} \text{ and } m = 0 \\ 2 \text{ solutions corresponding to } \pm m \neq 0 \\ \text{the remaining roots are complex} \end{cases} \\ T > T_c & \begin{cases} 2 \text{ solutions corresponding to } n = \frac{1}{2} \text{ and } m = 0 \\ \text{the remaining roots are complex} \end{cases} \end{cases}$$

- $z$  odd

$$\begin{cases} T < T_c & \begin{cases} 1 \text{ solution corresponding to } n = \frac{1}{2} \text{ and } m = 0 \\ 2 \text{ solutions corresponding to } \pm m \neq 0 \\ \text{the remaining roots are complex} \end{cases} \\ T > T_c & \begin{cases} 1 \text{ solution corresponding to } n = \frac{1}{2} \text{ and } m = 0 \\ \text{the remaining roots are complex} \end{cases} \end{cases}$$

By considering the following items: (i) the broken symmetry solution (i. e.,  $m \neq 0$ ) has a free energy lower than the one corresponding to the symmetric solution, (ii) the solution (136) is disregarded because not physical, (iii) the two solutions corresponding to  $\pm m$  are physically equivalent, (iv) all the complex solutions are disregarded, we can assert that the equation (131) admits only one solution of physical interest.

For  $T > T_c$  we have the following results:

$$\begin{aligned} X &= \frac{1}{2} [1 - \tanh(\beta J)] & n &= \frac{1}{2} & \lambda^{(1)} &= \frac{1}{4} [1 + \tanh(\beta J)] & p &= \tanh(\beta J) \\ m &= 0 & E &= -J \tanh(\beta J) & C &= k_B [\beta J \sec h(\beta J)]^2 & \chi &= \frac{\beta [1 + \tanh(\beta J)]}{1 - (z-1) \tanh(\beta J)}. \end{aligned} \quad (137)$$



For  $T < T_c$  the breaking symmetry solution depends on  $z$ . We shall present results for  $z = 3$  and  $z = 4$ . For the case of  $z = 3$  the critical temperature is given by  $\frac{k_B T_c}{J} = \frac{2}{\ln 3} \approx 1.82048$ . The solution of (131) is:

$$\begin{aligned} X &= \frac{(K+1)(K-2)+K\sqrt{(K+1)(K-3)}}{2(K^2-1)} \\ n &= \frac{(K+1)(K-2)+K\sqrt{(K+1)(K-3)}}{2(K+1)(K-2)} & m &= \frac{K\sqrt{(K+1)(K-3)}}{(K+1)(K-2)} \\ p &= \frac{1}{K-1} & \chi &= \frac{4\beta K}{(K-3)(K-2)^2(K+1)} \end{aligned} \quad (138)$$

For the case of  $z = 4$  the critical temperature is given by  $\frac{k_B T_c}{J} = \frac{2}{\ln 2} \approx 2.88539$ . The solution of (131) is:

$$\begin{aligned} X &= \frac{K^2-2+K\sqrt{K^2-4}}{2(K^2-1)} \\ n &= \frac{K^2-2+K\sqrt{K^2-4}}{2(K^2-2)} & m &= \frac{K\sqrt{K^2-4}}{K^2-2} \\ p &= \frac{1}{K^2-1} & \chi &= \frac{4\beta K^2}{(K^2-4)(K^2-2)^2} \end{aligned} \quad (139)$$

For all values of  $z$  the parameter  $\lambda^{(1)}$  and the internal energy  $E$  can be calculated by means of the expressions

$$\lambda^{(1)} = n[p + n(1-p)], \quad E = 4Jn(1-n)(1-p) - 2J. \quad (140)$$

In Fig. 1 we plot the magnetization per site  $m = \langle S(0) \rangle$  as a function of the temperature, expressed in units of  $J$ , for the values of the coordination number  $z = 3$  and  $z = 4$ . As expected, the magnetization decreases by increasing  $T$  and vanishes at the critical temperature  $T_c$ , determined by (135). By expanding the parameter  $K = e^{2\beta J}$  around the critical temperature  $T_c$ :

$$K = K_c [1 + bt] + O(t^2) \quad (141)$$

where

$$t = \frac{T_c - T}{T_c}, \quad K_c = \frac{z}{z-2}, \quad b = \ln\left(\frac{z}{z-2}\right), \quad (142)$$

we can easily show that close to  $T_c$  the magnetization behaves as

$$m = \begin{cases} \frac{3\sqrt{3bt}}{2} + O(t^{\frac{3}{2}}) & z = 3 \\ 2\sqrt{2bt} + O(t^{\frac{3}{2}}) & z = 4 \end{cases} \quad (143)$$

with critical exponents  $\beta = \frac{1}{2}$ , in agreement with Refs. [25, 26]. The behaviour of the parameter  $p = 1 - \frac{\langle n^\alpha(0) \rangle_0}{\langle n(0) \rangle}$  as a function of  $\frac{T}{J}$  is shown in Fig. 2. By increasing the temperature,  $p$  first increases up to the maximum value  $(p)_{T=T_c} = \frac{1}{z-1}$ , then decreases. It is always  $p < 1$ : this condition implies the ergodic behaviour of the spin correlation functions, when we recall the results of Section 6. We notice that for  $T > T_c$  the value of  $p$  is the same for all values of  $z$ .

In Fig. 3 we report the temperature dependence of the spin susceptibility per site  $\chi = \left(\frac{\partial m}{\partial h}\right)_{h=0}$ . This quantity diverges at  $T = T_c$  with critical exponents  $\gamma = \gamma' = -1$ :

$$\chi = \begin{cases} \frac{(T_c - T)^{-1}}{b(z-2)} & T < T_c \\ \frac{2(T - T_c)^{-1}}{b(z-2)} & T > T_c \end{cases} \quad (144)$$

To the contrary of  $p$ , above  $T_c$  the susceptibility changes with  $z$ , as can be seen by Eq. (78). The specific heat per site is reported in Fig. 4 as a function of the temperature. We observe a jump in correspondence of  $T_c$ , with critical exponents  $\alpha = 0$ , as expected for a second order phase transition. It can be shown that the jump  $\Delta C$  at  $T = T_c$  is given by:

$$\Delta C = \begin{cases} \frac{21J^2}{4T_c^2} - \frac{4J^2K_c}{(1+K_c)^2T_c^2} = \frac{9}{8} [\ln 3]^2 & z = 3 \\ \frac{80J^2}{9T_c^2} - \frac{4J^2K_c}{(1+K_c)^2T_c^2} = 2 [\ln 2]^2 & z = 4 \end{cases} \quad (145)$$

We notice that the jump decreases with  $z$ , and that above  $T_c$  the behaviour of the specific heat does not depend on the value of  $z$ .

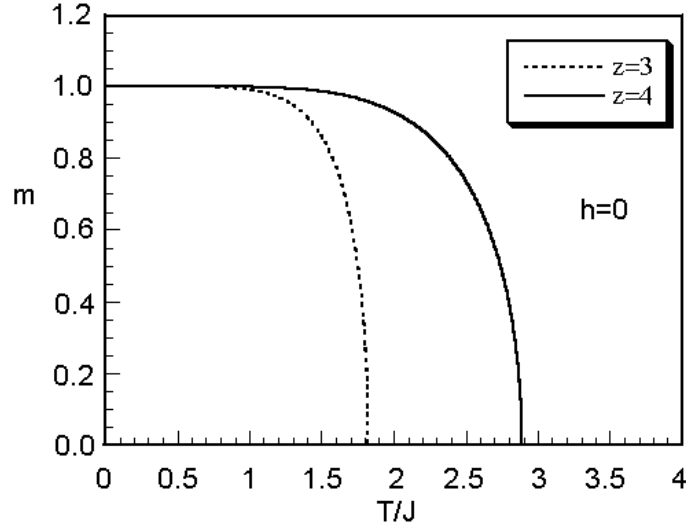


FIG. 1: The magnetization  $m$  is plotted against  $T/J$  for  $z = 3$  and  $z = 4$  and zero magnetic field.

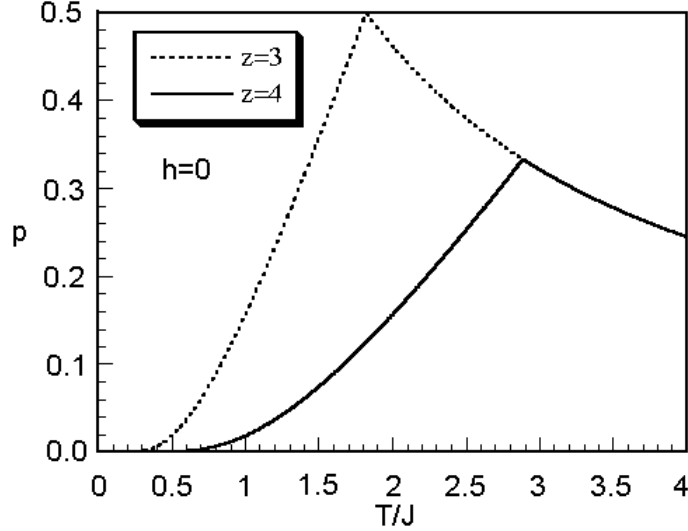


FIG. 2: The parameter  $p$  is plotted against  $T/J$  for  $z = 3$  and  $z = 4$  and zero magnetic field.

The temperature dependence of the internal energy  $E$ , of the free energy  $F$  and of the entropy  $S$  is shown in Figs. 5, 6 and 7, respectively. We observe the different behaviour at  $T_c$ :  $F$  is a smooth function, while  $E$  and  $S$  exhibit a drastic change. This behaviour shows that at  $T_c$  we have a second-order phase transition. Also, we note that above  $T_c$  the internal energy does not depend on  $z$ , while the free energy and the entropy depend on  $z$ .

In Fig. 8 we plot the spin correlation function  $\Lambda^{(0)}(j) = \langle n(0)n(j) \rangle$  versus the distance  $j$  for  $z = 3$  and several values of the temperature, chosen below and above the critical temperature. We clearly see that a long-range ferromagnetic order is established below  $T_c$ .

### B. Finite magnetic field

For finite magnetic field the equation (62) for the parameter  $X$  can be written as

$$(1 - X) H [1 + (K^2 - 1) X]^{z-1} - X K^z = 0 \quad (146)$$

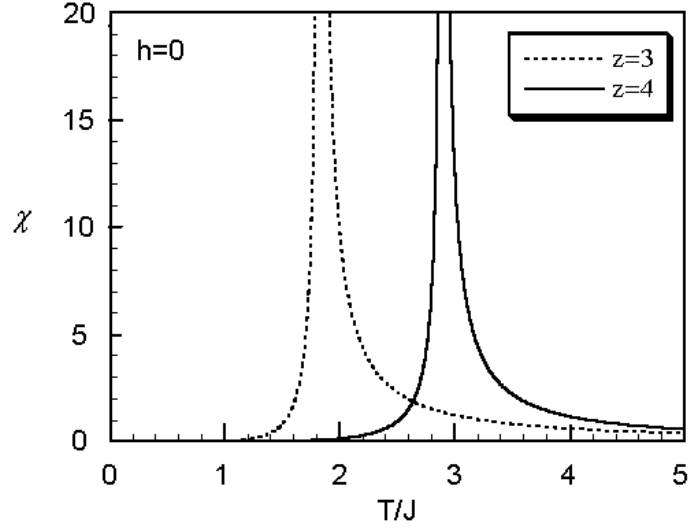


FIG. 3: The susceptibility  $\chi$  is plotted against  $T/J$  for  $z = 3$  and  $z = 4$  and zero magnetic field.

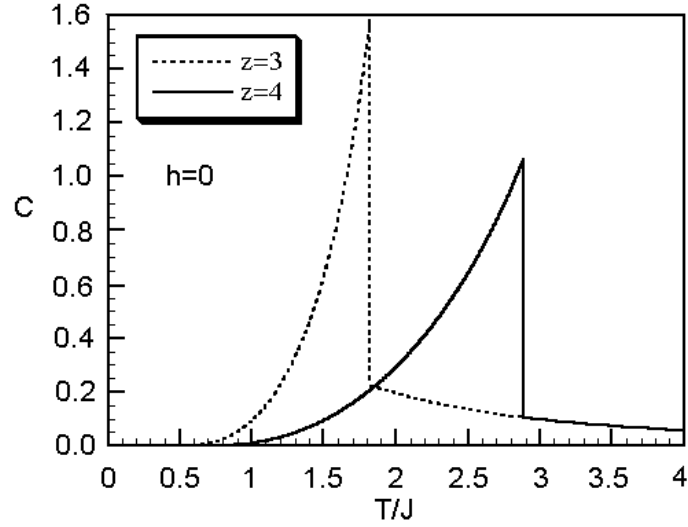


FIG. 4: The specific heat  $C$  is plotted against  $T/J$  for  $z = 3$  and  $z = 4$  and zero magnetic field.

where we put  $K = e^{2\beta J}$  and  $H = e^{2\beta h}$ . For  $H \neq 1$  this equation does not admit a general solution for any value of the coordination number, and we must discuss case by case. For  $z = 2$  the solution is

$$X = \frac{K^2(H-1) - 2H + K\sqrt{K^2 + H^2K^2 - 2H(K^2 - 2)}}{2H(K^2 - 1)} \quad (147)$$

which describes the well known solution of the one-dimensional spin- $\frac{1}{2}$  Ising model. The other root of Eq. (146) corresponds to a physically unstable system and is disregarded.

For  $z = 3$  it is possible to show that there is a critical temperature  $T_c(h)$ , depending on the magnetic field, such that for  $T < T_c(h)$  there are three real and unequal roots, while for  $T > T_c(h)$  there is one real root and two conjugate imaginary roots.  $T_c(h)$  is determined by the following equation:

$$8K^3H = K^4 + 18K^2 - 27 - (K^2 - 9)^{3/2} \sqrt{K^2 - 1}. \quad (148)$$

For  $z = 4$  there are four real and unequal roots for  $T < T_c(h)$ , while for  $T > T_c(h)$  there are two real unequal roots and two conjugate imaginary roots.

In Fig. 9 we report  $T_c(h)$  as a function of the magnetic field for  $z = 3, 4$ . By increasing  $|h|$ ,  $T_c(h)$  decreases from the value  $k_B T_c = \frac{2J}{\ln(\frac{z}{z-2})}$  [cfr. Eq. (135)] and vanishes at  $|h| = J(z-2)$ .

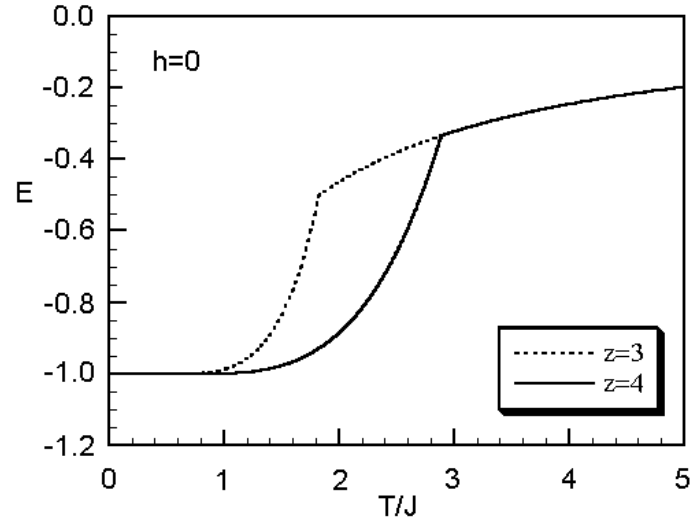


FIG. 5: The internal energy  $E$  is plotted against  $T/J$  for  $z = 3$  and  $z = 4$  and zero magnetic field.

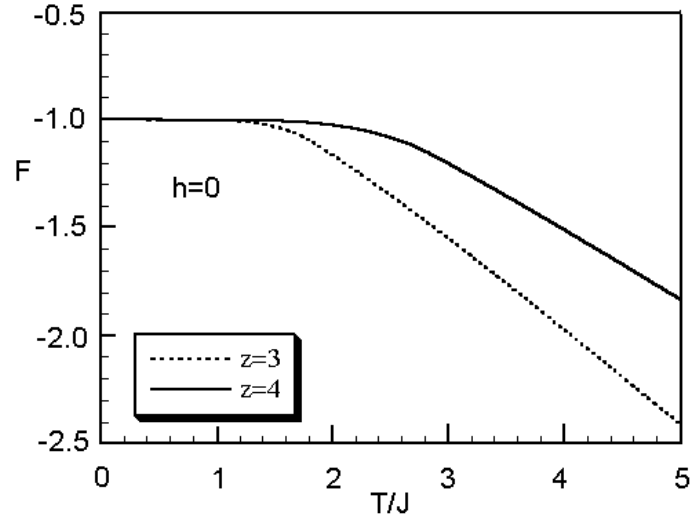


FIG. 6: The free energy  $F$  is plotted against  $T/J$  for  $z = 3$  and  $z = 4$  and zero magnetic field.

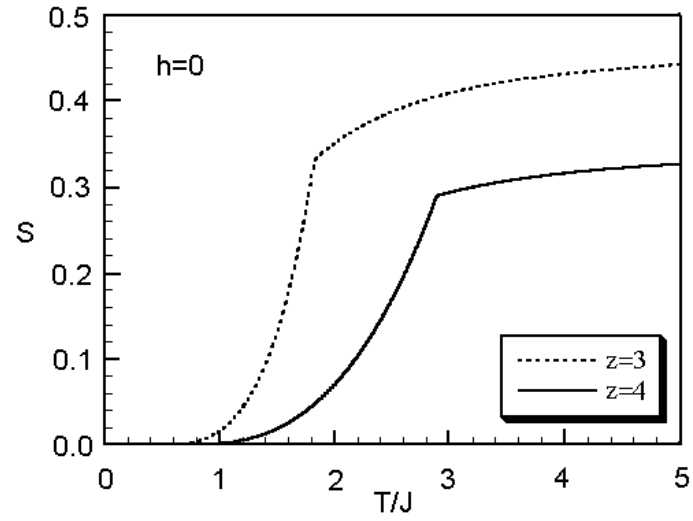


FIG. 7: The entropy  $S$  is plotted against  $T/J$  for  $z = 3$  and  $z = 4$  and zero magnetic field.

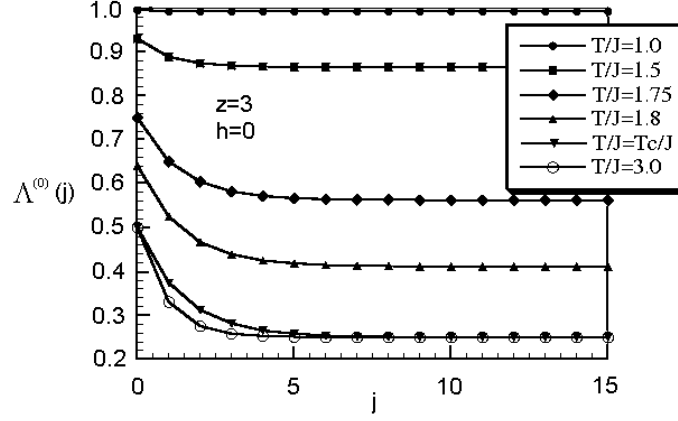


FIG. 8: The spin correlation function  $\Lambda^{(0)}(j)$  is plotted against  $j$  for  $z = 3$  and zero magnetic field.

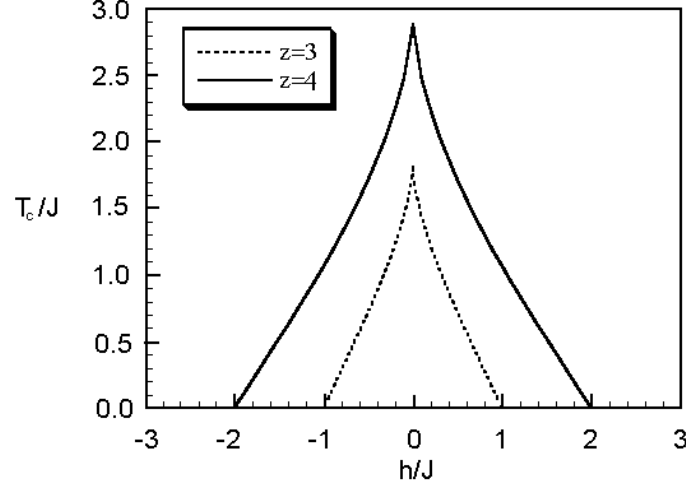


FIG. 9: The temperature  $T_c(h)$  is plotted against the magnetic field  $h$  for  $z = 3, 4$ .

For  $z = 3$ , below  $T_c(h)$  the three real solutions have the following behavior: one solution satisfies the particle-hole symmetry (i. e.  $\langle n \rangle = \frac{1}{2}$  at  $\mu = -2zJ$ ) but corresponds to an unstable system (for example the compressibility is negative). The other two solutions violate the particle-hole symmetry and describe a finite magnetization for any value of the magnetic field, in a direction parallel and anti-parallel to  $h$ , respectively. By disregarding the unstable solution and by picking up the one which describes a magnetization in the direction of the magnetic field, we can assert that in the entire plane  $(h, T)$  Eq. (146) admits only one physical solution given by

$$X = 2\sqrt{\frac{a}{3}} \cos(\theta) - \frac{c}{3} \quad T < T_c(h)$$

$$X = \sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}} + \sqrt[3]{\frac{b}{2} - \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}} - \frac{c}{3} \quad T > T_c(h) \quad (149)$$

where

$$\begin{aligned} a &= \frac{K^3(KH-3)}{3H(K^2-1)^2} & b &= \frac{K^3(2HK^3-9K^2+27)}{27H(K^2-1)^3} \\ c &= \frac{3-K^2}{K^2-1} & \theta &= \frac{1}{3} \cos^{-1} \left( \frac{3^{3/2}b}{2a^{3/2}} \right) \end{aligned} \quad (150)$$

Similar situation holds for  $z = 4$ . Below  $T_c(h)$ , two solutions satisfy the particle-hole symmetry but correspond to an unstable system; the other two solutions violate the particle-hole symmetry and describe a finite magnetization, parallel and anti-parallel to  $h$ , respectively. Above  $T_c(h)$ , among the two solutions, only one has a physical meaning.

Once  $X$  is known, we can calculate the various properties by using the formulas given in Sections 5 and 6. The behaviour of the magnetization, the susceptibility and the specific heat as functions of the temperature is reported in Figs. 10, 12 and 13, respectively, for several values of the magnetic field. At low temperatures the system is fully polarized by any finite magnetic field. By increasing temperature, the magnetization decreases and tends to zero in the limit  $T \rightarrow \infty$ . The different behavior of the magnetization, below and above the critical temperature  $T_c$ , is shown in Fig. 11, where  $m$  is plotted as a function of the magnetic field for several values of  $T$ . For finite  $h$  the susceptibility and the specific heat do not exhibit a discontinuity, there is a peak at a certain temperature  $T^*$  which increases with  $h$ .

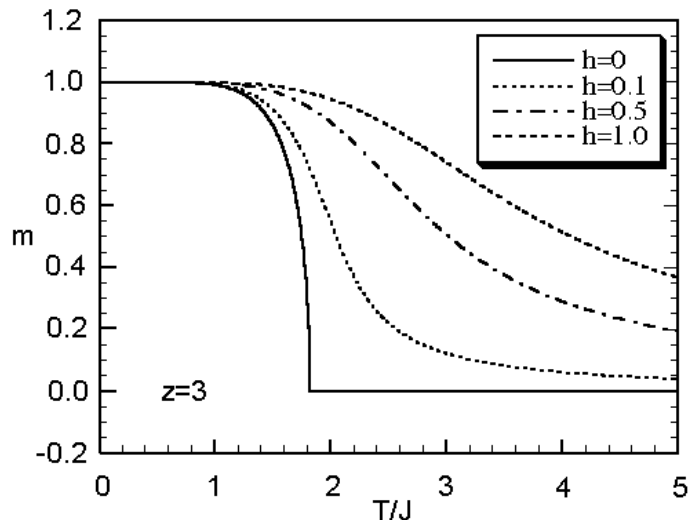


FIG. 10: The magnetization  $m$  is plotted against  $T/J$  for  $z = 3$  and several values of the magnetic field.

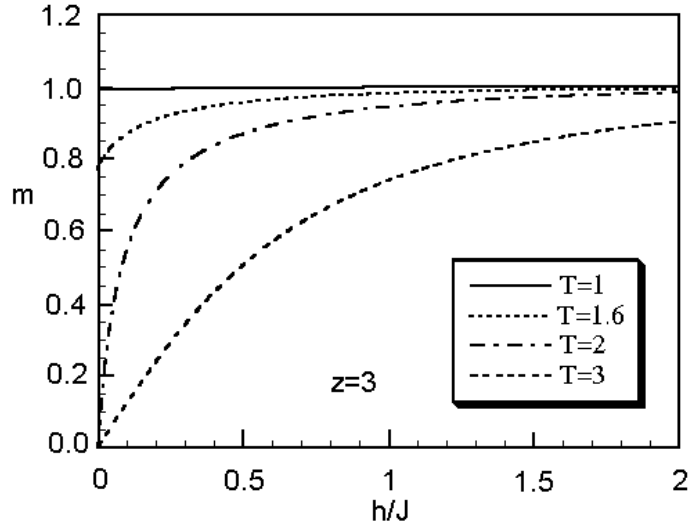


FIG. 11: The magnetization  $m$  is plotted against  $h/J$  for  $z = 3$  and several values of the temperature.

### VIII. CONCLUSIONS

In this paper we have studied the ferromagnetic spin- $\frac{1}{2}$  Ising model in the presence of an external magnetic field on the Bethe lattice by means of the equations of motion method and of the Green's function formalism. First, it has been shown that there exists a mapping between such a model and a fermionic one built of one species of particles localized on the sites of a Bethe lattice and interacting via an intersite Coulomb interaction. Then, an exact

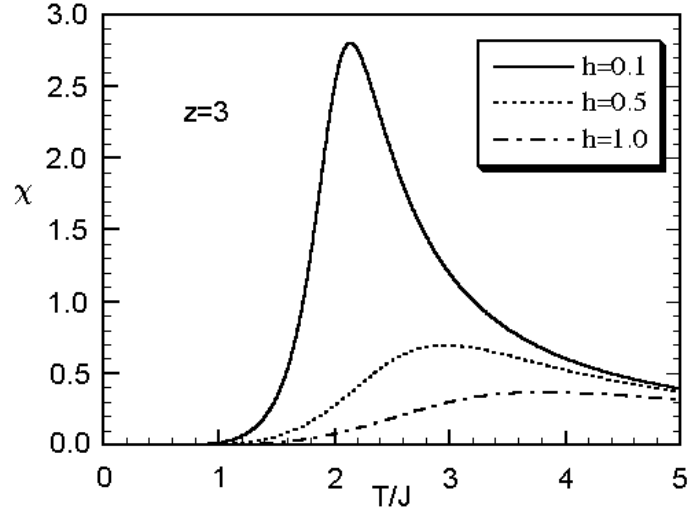


FIG. 12: The susceptibility  $\chi$  is plotted against  $T/J$  for  $z = 3$  and several values of the magnetic field.

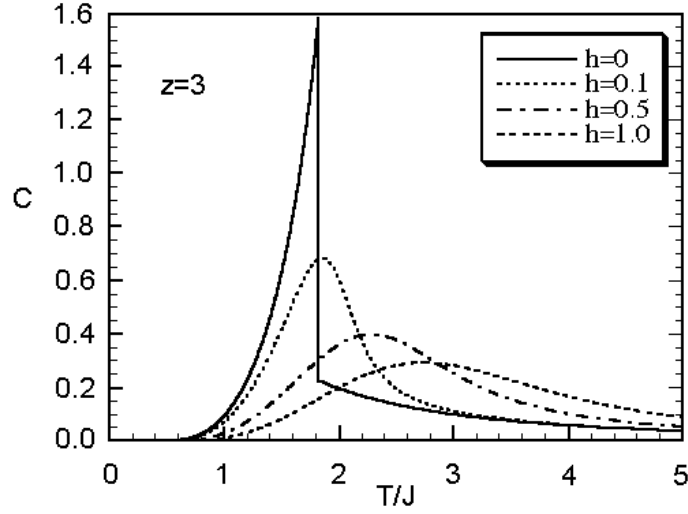


FIG. 13: The specific heat  $C$  is plotted against  $T/J$  for  $z = 3$  and several values of the magnetic field.

solution has been found for the Bethe lattice with any coordination number  $z$  in terms of a finite and complete set of eigenoperators and eigenvalues of the fermionic Hamiltonian. This solution allowed us to write exact expressions for the corresponding Green's function and correlation functions which depend on a finite set of parameters to be determined in a self-consistent way. Such parameters have been exactly fixed by means of algebra constraints. So local and non local correlation functions have been calculated up to four point ones, together with the corresponding physical quantities, i.e. the particle density, the magnetization per site, the susceptibility, the correlation length, the internal energy per site, the specific heat and the entropy. All the results are in perfect agreement with the ones existing in literature. The results are discussed in great detail for the first two values  $z = 3, 4$  of the coordination number of the Bethe lattice with and without external magnetic field. Our procedure allows also to generalize the known results, as it has been explicitly shown in the case of non local correlation functions, where new expressions, no previously reported, for the three-point spin-spin correlation function have been obtained, together with a general calculation scheme which can give rise to higher order correlators.

#### Acknowledgments

One of the authors (F. M.) wishes to thank Professors G. Baskaran and N. M. Plakida for stimulating correspondence on the problem of the Ising model. The authors would like also to thank dr. A. Avella for stimulating discussions

and for a careful reading of the manuscript.

### APPENDIX A: CALCULATION OF $[n^\alpha]^k$

Let us recall the following definition

$$n^\alpha(i) = \frac{1}{z} (n_1 + n_2 + \dots + n_z) \quad (\text{A1})$$

where  $n_p$  ( $p = 1, \dots, z$ ) are the first nearest neighbors of the site  $i$ . Then we have to calculate the power

$$[n^\alpha(i)]^k = \frac{1}{z^k} (n_1 + n_2 + \dots + n_z)^k. \quad (\text{A2})$$

By considering the algebraic property  $[n_p]^m = n_p$  ( $m = 1, 2, \dots$ ), after some easy but lengthy calculations, it can be shown that

$$[n^\alpha(i)]^k = \frac{1}{z^k} \sum_{p=1}^z b_p^{(k)} N_p^{(z)} \quad (\text{A3})$$

where  $N_p^{(z)}$  are the operators

$$N_p^{(z)} = \sum_{l_1 < l_2 < \dots < l_p=1}^z n_{l_1} n_{l_2} \dots n_{l_p} \quad (\text{A4})$$

and  $b_p^{(k)}$  are some positive integer numbers, defined as

$$b_1^{(k)} = 1$$

$$b_p^{(k)} = \sum_{l_1=p-1}^{k-1} \sum_{l_2=p-2}^{l_1-1} \sum_{l_3=p-3}^{l_2-1} \dots \sum_{l_{p-2}=2}^{l_{p-3}-1} \sum_{l_{p-1}=1}^{l_{p-2}-1} \binom{k}{l_1} \binom{l_1}{l_2} \binom{l_2}{l_3} \dots \binom{l_{p-3}}{l_{p-2}} \binom{l_{p-2}}{l_{p-1}} \quad (p > 1). \quad (\text{A5})$$

It is important to notice that  $b_p^{(k)} = 0$  for  $p > k$ . In particular for the first values of  $p$  we get:

$$\begin{aligned} b_2^{(n)} &= 2(2^{n-1} - 1) \\ b_3^{(n)} &= 3(3^{n-1} - 2^n + 1) \\ b_4^{(n)} &= 4(4^{n-1} - 3^n + 3 \cdot 2^{n-1} - 1) \\ b_5^{(n)} &= 5(5^{n-1} - 4^n + 2 \cdot 3^n - 2^{n+1} + 1) \end{aligned} \quad (\text{A6})$$

The above results are valid for any lattice with coordination number  $z$ .

### APPENDIX B: CALCULATION OF THE COEFFICIENTS $A_m^{(k)}$

Given the results in Appendix A, we can write

$$[n^\alpha(i)]^k = \sum_{m=1}^z A_m^{(k)} [n^\alpha(i)]^m \quad (\text{B1})$$

where the coefficients  $A_m^{(k)}$  are some rational numbers which must satisfy the relations

$$\begin{aligned} \sum_{m=1}^z A_m^{(k)} &= 1 \\ A_m^{(k)} &= \delta_{m,k} \quad (k = 1, \dots, z) \end{aligned} \quad (\text{B2})$$

The first relation follows by putting  $n^\alpha(i) = 1$  while the second can be derived by considering the case  $1 \leq k \leq z$ . Indeed we must calculate the coefficients  $A_m^{(k)}$  only for  $k \geq z+1$  and  $m = 1, \dots, z$ . By noting that for  $k \geq z+2$  we can write

$$\sum_{m=1}^z A_m^{(k)} [n^\alpha(i)]^m = \sum_{m=1}^z A_m^{(k-1)} [n^\alpha(i)]^{m+1}, \quad (\text{B3})$$



the following recursion rule can be established

$$A_m^{(k)} = A_{m-1}^{(k-1)} + A_z^{(k-1)} A_m^{(z+1)} \quad (m = 1, \dots, z) \quad A_0^{(k-1)} = 0. \quad (\text{B4})$$

This rule implies that we must calculate only the  $z$  coefficients  $A_m^{(z+1)}$ , ( $m = 1, \dots, z$ ), which can be done by means of the relation (B1) evaluated for  $k = z + 1$ :

$$[n^\alpha(i)]^{z+1} = \sum_{m=1}^z A_m^{(z+1)} [n^\alpha(i)]^m. \quad (\text{B5})$$

By using the results given in Appendix A to rewrite (B5), we obtain the equation

$$\sum_{k=1}^z b_k^{(z+1)} N_k^{(z)} = \sum_{m=1}^z A_m^{(z+1)} z^{z+1-m} \sum_{k=1}^m b_k^{(m)} N_k^{(z)}, \quad (\text{B6})$$

which, by noting that the operators  $N_k^{(z)}$  are linearly independent, takes the form

$$\sum_{m=k}^z A_m^{(z+1)} z^{z+1-m} b_k^{(m)} - b_k^{(z+1)} = 0 \quad (k = 1, \dots, z). \quad (\text{B7})$$

Such equations give rise to the iterative solution

$$\begin{aligned} A_z^{(z+1)} &= \frac{b_z^{(z+1)}}{z b_z^{(z)}} \\ A_{z-1}^{(z+1)} &= \frac{1}{z^2 b_{z-1}^{(z-1)}} \left[ b_{z-1}^{(z+1)} - A_z^{(z+1)} z b_{z-1}^{(z)} \right] \\ A_{z-2}^{(z+1)} &= \frac{1}{z^3 b_{z-2}^{(z-2)}} \left[ b_{z-2}^{(z+1)} - A_{z-1}^{(z+1)} z^2 b_{z-2}^{(z-1)} - A_z^{(z+1)} z b_{z-2}^{(z)} \right] \\ &\vdots \\ A_1^{(z+1)} &= \frac{1}{z^z} \left[ 1 - A_2^{(z+1)} z^{z-1} - A_3^{(z+1)} z^{z-2} - \dots - A_{z-2}^{(z+1)} z^3 - A_{z-1}^{(z+1)} z^2 - A_z^{(z+1)} z \right] \end{aligned} \quad (\text{B8})$$

As an example, we give the values of the coefficients  $A_m^{(k)}$  for the first values of  $z$

$$\begin{aligned} z=2: & \quad A_1^{(3)} = -\frac{1}{2} \quad A_2^{(3)} = \frac{3}{2} \\ z=3: & \quad A_1^{(4)} = \frac{2}{9} \quad A_2^{(4)} = -\frac{11}{9} \quad A_3^{(4)} = 2 \\ z=4: & \quad A_1^{(5)} = -\frac{3}{32} \quad A_2^{(5)} = \frac{25}{32} \quad A_3^{(5)} = -\frac{35}{16} \quad A_4^{(5)} = \frac{5}{2} \\ z=5: & \quad A_1^{(6)} = \frac{24}{625} \quad A_2^{(6)} = -\frac{274}{625} \quad A_3^{(6)} = \frac{9}{5} \quad A_4^{(6)} = -\frac{17}{5} \quad A_5^{(6)} = 3 \\ z=6: & \quad A_1^{(7)} = -\frac{5}{324} \quad A_2^{(7)} = \frac{49}{216} \quad A_3^{(7)} = -\frac{203}{162} \quad A_4^{(7)} = \frac{245}{72} \quad A_5^{(7)} = -\frac{175}{36} \quad A_6^{(7)} = \frac{7}{2} \end{aligned} \quad (\text{B9})$$

### APPENDIX C: CALCULATION OF $\langle [n^\alpha(0)]^k \rangle_0$

By means of the results given in Appendix A we can write that:

$$\langle [n^\alpha(0)]^k \rangle_0 = \frac{1}{z^k} \sum_{p=1}^z b_p^{(k)} \langle N_p^{(z)} \rangle_0. \quad (\text{C1})$$

The particular topology of the Bethe lattice allows us to decouple the correlation functions in the  $H_0$ -representation (see Eq. (48)) and to obtain

$$\langle N_p^{(z)} \rangle_0 = \sum_{l_1 < l_2 < \dots < l_p=1}^z \langle n_{l_1} n_{l_2} \dots n_{l_p} \rangle_0 = [\langle n^\alpha(0) \rangle_0]^p \sum_{l_1 < l_2 < \dots < l_p=1}^z = \binom{z}{p} [\langle n^\alpha(0) \rangle_0]^p. \quad (\text{C2})$$

It follows that

$$\langle [n^\alpha(0)]^k \rangle_0 = \frac{1}{z^k} \sum_{p=1}^z \binom{z}{p} b_p^{(k)} X^p \quad (\text{C3})$$

where  $X = \langle n^\alpha(0) \rangle_0$  (see Eq. (51)).

# APPENDIX D: CALCULATION OF $\lambda^{(k)}$ AND $\kappa^{(k)}$

In this Appendix we will calculate the local correlators  $\lambda^{(k)}$  and  $\kappa^{(k)}$  in terms of the parameter  $X$  defined in Eq. (51). We recall the definitions given in Section 5:

$$\begin{aligned}\lambda^{(k)} &= \left\langle n(0) [n^\alpha(0)]^k \right\rangle \\ \kappa^{(k)} &= \left\langle [n^\alpha(0)]^k \right\rangle.\end{aligned}\tag{D1}$$

Let us start with  $\lambda^{(k)}$  which, according to (42), can be written as

$$\lambda^{(k)} = \frac{\left\langle n(0) [n^\alpha(0)]^k e^{-\beta H_I} \right\rangle_0}{\left\langle e^{-\beta H_I} \right\rangle_0}.\tag{D2}$$

By recalling Eq. (69) and the recursion rule (A3), we have

$$\left\langle n(0) [n^\alpha(0)]^k e^{-\beta H_I} \right\rangle_0 = \frac{1}{z^k} \sum_{p=1}^z b_p^{(k)} \langle n(0) \rangle_0 \left\langle N_p^{(z)} \prod_{i=1}^z [1 + A n_i] \right\rangle_0.\tag{D3}$$

Furthermore, by recalling Eq. (A4) it is immediate to show that

$$\left\langle N_p^{(z)} \prod_{i=1}^z [1 + A n_i] \right\rangle_0 = \binom{z}{p} (1+A)^p X^p (1+AX)^{z-p}.\tag{D4}$$

By putting (D3) and (D4) into (D2) and recalling the results (68) and (70) we obtain for  $k \geq 1$

$$\lambda^{(k)} = \frac{(1-X)}{1+AX^2} e^{\beta\mu} \frac{1}{z^k} \sum_{p=1}^z \binom{z}{p} b_p^{(k)} (1+A)^p X^p (1+AX)^{z-p}.\tag{D5}$$

Use of the equation (62) for the parameter  $X$  allows us to rewrite (D5) under the form

$$\lambda^{(k)} = \frac{1}{1+AX^2} \sum_{p=1}^k a_p^{(z,k)} \frac{X^{p+1} (1+A)^p}{(1+AX)^{p-1}}\tag{D6}$$

where

$$a_p^{(z,k)} = \frac{1}{z^k} b_p^{(k)} \binom{z}{p}.\tag{D7}$$

In order to calculate  $\kappa^{(k)}$ , let us start from the equation:

$$\kappa^{(k)} = C_{1,k+1} + \lambda^{(k)}.\tag{D8}$$

By putting together (74) and (D6), we obtain:

$$\kappa^{(k)} = \frac{1}{1+AX^2} \sum_{p=1}^z a_p^{(z,k)} X^p \left[ (1-X) + \frac{X(1+A)^p}{(1+AX)^{p-1}} \right].\tag{D9}$$

In particular, from (D6) and (D9) we have

$$\begin{aligned}\lambda^{(1)} &= \frac{X^2(1+A)}{1+AX^2} \\ \lambda^{(2)} &= \frac{1}{z} \lambda^{(1)} + \frac{z-1}{z} \frac{X^3(1+A)^2}{(1+AX^2)(1+AX)}\end{aligned}\tag{D10}$$

and

$$\begin{aligned}\kappa^{(1)} &= \frac{X(1+AX)}{1+AX^2} \\ \kappa^{(2)} &= \frac{1}{z}\kappa^{(1)} + \frac{z-1}{z} \frac{X^2}{1+AX^2} \left[ (1-X) + \frac{X(1+A)^2}{(1+AX)} \right].\end{aligned}\quad (\text{D11})$$

In closing this Appendix, we note the following useful relations

$$\begin{aligned}\kappa^{(2)} &= \frac{1}{z}n + \frac{z-1}{z} \left[ n^2 + \frac{(n^2 - \lambda^{(1)})^2}{n(1-n)} \right] \\ \kappa^{(2)} - \lambda^{(2)} &= \frac{1}{z} \left( n - \lambda^{(1)} \right) + \frac{z-1}{z} \frac{(n - \lambda^{(1)})^2}{(1-n)} \\ \lambda^{(2)} &= \frac{1}{z}\lambda^{(1)} + \frac{z-1}{z} \frac{[\lambda^{(1)}]^2}{n}.\end{aligned}\quad (\text{D12})$$

where we used  $n = \kappa^{(1)}$ .

### APPENDIX E: CALCULATION OF $\Lambda^{(0)}(j)$ AND $K^{(1)}(j)$

Let us recall the following definitions given in Section 6:

$$\begin{aligned}K^{(k)}(j) &= \left\langle [n^\alpha(0)]^k n(j) \right\rangle \\ \Lambda^{(k)}(j) &= \left\langle n(0) [n^\alpha(0)]^k n(j) \right\rangle\end{aligned}\quad (\text{E1})$$

where  $j$  is a site at a distance of  $j$  ( $j \geq 2$ ) steps from the central site. Let us make for simplicity the choice that  $j$  belongs to the  $z$ -th subtree (but any subtree can be chosen) and let us define the correlation function of the composite fields as

$$D^{(k)}(j) = \left\langle c(0) c^\dagger(0) [n^\alpha(0)]^{k-1} n(j) \right\rangle \quad (k \geq 1). \quad (\text{E2})$$

By means of the commutation relations we note that

$$D^{(k)}(j) = K^{(k-1)}(j) - \Lambda^{(k-1)}(j), \quad (\text{E3})$$

while, by using (42) and the algebraic relation (46), we have

$$D^{(k)}(j) = \frac{\left\langle c(0) c^\dagger(0) [n^\alpha(0)]^{k-1} n(j) \right\rangle_0}{\left\langle e^{-\beta H_I} \right\rangle_0}. \quad (\text{E4})$$

By means of the properties of the correlation functions in the  $H_0$ -representation (see Eq. (48)) we get:

$$\begin{aligned}D^{(k)}(j) &= \frac{\left\langle c(0) c^\dagger(0) \right\rangle_0}{\left\langle e^{-\beta H_I} \right\rangle_0} \left\langle [n^\alpha(0)]^{k-1} [n(j)] \right\rangle_0 \\ &= C_{1,1} \left\langle [n^\alpha(0)]^{k-1} [n(j)] \right\rangle_0 = (1-n) \left\langle [n^\alpha(0)]^{k-1} [n(j)] \right\rangle_0,\end{aligned}\quad (\text{E5})$$

where we used Eq. (53). By putting together (E3) and (E5) we obtain

$$K^{(k-1)}(j) - \Lambda^{(k-1)}(j) = (1-n) \left\langle [n^\alpha(0)]^{k-1} [n(j)] \right\rangle_0, \quad (\text{E6})$$

which in particular for  $k = 1$  reads

$$n - \Lambda^{(0)}(j) = (1-n) \left\langle n(j) \right\rangle_0. \quad (\text{E7})$$

Let us notice that  $\langle n(j) \rangle_0$  depends on the site  $j$  because the  $H_0$ -representation lacks of translational invariance. Let us now start to calculate the function  $\Lambda^{(0)}(j) = \langle n(0) n(j) \rangle$ , which in the  $H_0$ -representation can be written as

$$\Lambda^{(0)}(j) = \langle n(0) n(j) \rangle = \frac{\langle n(0) n(j) e^{-\beta H_I} \rangle_0}{\langle e^{-\beta H_I} \rangle_0}. \quad (\text{E8})$$

By recalling that (cfr. (69))  $e^{-\beta H_I} = \prod_{i=1}^z [1 + A n(0) n_i]$  we have

$$\langle n(0) n(j) e^{-\beta H_I} \rangle_0 = \langle n(0) \rangle_0 (1 + AX)^{z-1} [\langle n(j) \rangle_0 + A \langle n_z n(j) \rangle_0]. \quad (\text{E9})$$

Putting (E9) into (E8), recalling the results (68) and (70), using the equation (62) for the parameter  $X$ , we obtain

$$\Lambda^{(0)}(j) = \frac{X}{1 + AX^2} [\langle n(j) \rangle_0 + A \langle n_z n(j) \rangle_0]. \quad (\text{E10})$$

By combining (E7) and (E10) we can express the unknown correlation functions  $\langle n(j) \rangle_0$  and  $\langle n_z n(j) \rangle_0$  in terms of the two-point correlation function  $\Lambda^{(0)}(j)$  as follows

$$\begin{aligned} \langle n(j) \rangle_0 &= \frac{1}{(1-n)} [n - \Lambda^{(0)}(j)] \\ \langle n_z n(j) \rangle_0 &= \left[ \frac{1 + AX^2}{AX} + \frac{1}{A} \frac{1}{(1-n)} \right] \Lambda^{(0)}(j) - \frac{1}{A} \frac{n}{(1-n)}. \end{aligned} \quad (\text{E11})$$

Let us now calculate the function  $\Lambda^{(1)}(j) = \langle n(0) n^\alpha(0) n(j) \rangle$  by following the same procedure we adopted for the calculation of  $\Lambda^{(0)}(j)$ . Recalling (68) and (70), and the basic equation (62) we get

$$\begin{aligned} \Lambda^{(1)}(j) &= \frac{(z-1)}{z} \frac{X^2(1+A)}{(1+AX)(1+AX^2)} [\langle n(j) \rangle_0 + A \langle n_z n(j) \rangle_0] \\ &\quad + \frac{1}{z} \frac{X(1+A)}{1+AX^2} \langle n_z n(j) \rangle_0. \end{aligned} \quad (\text{E12})$$

Now, by using the relations in (E11) it is possible to express the correlation function  $\Lambda^{(1)}(j)$  in terms of  $\Lambda^{(0)}(j)$ :

$$\Lambda^{(1)}(j) = -\frac{1}{z} \frac{X(1+A)}{1+AX^2} \frac{1}{A} \frac{n}{(1-n)} + \frac{(z-1)}{z} \frac{X(1+A)}{(1+AX)} \Lambda^{(0)}(j) + \frac{1}{z} \frac{(1+A)}{A(1-X)} \Lambda^{(0)}(j). \quad (\text{E13})$$

By using the relation (E6) for  $k=2$  we are now in position to calculate the function  $K^{(1)}(j)$  as

$$K^{(1)}(j) = \Lambda^{(1)}(j) + (1-n) \langle n^\alpha(0) n(j) \rangle_0, \quad (\text{E14})$$

which, by observing that

$$\langle n^\alpha(0) n(j) \rangle_0 = \frac{1}{z} \langle n_z n(j) \rangle_0 + \frac{z-1}{z} X \langle n(j) \rangle_0, \quad (\text{E15})$$

and recalling (E11) and (E13), becomes

$$K^{(1)}(j) = \frac{1}{zp} [\Lambda^{(0)}(j) - n^2] + \frac{(z-1)p}{z} [\Lambda^{(0)}(j) - n^2] + n^2 \quad (\text{E16})$$

where we made use of the relations (72) and (79).

Recalling now the recursion relation (cfr. (89))

$$K^{(1)}(j) = \frac{1}{z} \Lambda^{(0)}(j-1) + \frac{z-1}{z} \Lambda^{(0)}(j+1) \quad (\text{E17})$$

and putting that together with (E16), we finally obtain the relevant recurrence relation

$$G(j+1) - pG(j) = \frac{1}{p(z-1)} [G(j) - pG(j-1)] \quad (\text{E18})$$

where we defined

$$G(j) = \Lambda^{(0)}(j) - n^2 = \langle n(0)n(j) \rangle - n^2. \quad (\text{E19})$$

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